1 Homework set 1, due Jan 25

1. **Plane wave point particle**

   Consider a relativistic point particle in a particle in a plane-wave space-time, whose metric is of the form

   $$ds^2 = -dx^+ dx^- - \beta x_\perp^2 (dx^+)^2 + dx_\perp^2 \tag{1}$$

   (a) Use the reparametrization invariance of the action to choose light-cone time $x^+ = \tau$ and show that the dynamics of the particle can be solved exactly in terms of harmonic oscillators.

   (b) Solve the dynamics explicitly in the lightcone parametrization.

   (c) Introduce a Lagrange multiplier to eliminate the square root of the action (just like in class). Calculate the equations of motion of the point particle/

   (d) Choose $\eta = 1$ and solve the system. To do this notice that the problem is separable. Compare your answers with (b) to show that you get equivalent results.

2. **Lorentz symmetry**

   (a) Find the Noether currents on the worldsheet associated to Lorentz transformations.

   (b) Write an expression for the angular momentum of the string in static gauge.

3. **Folded string solution**

   (a) Show that it's possible to find solutions of the string equations of the following form:

   $$x^0 = Et \tag{2}$$
   $$x^1 = C \sin\sigma \cos t \tag{3}$$
   $$x^2 = D \sin\sigma \sin t \tag{4}$$

   This problem requires finding the relations between C, D, E.

   (b) Calculate the energy of the string configuration.

   (c) Calculate the spin (angular momentum) of the configuration.

(d) Show that the string energy (mass) squared is proportional to the angular momentum. This constant of proportionality is called the Regge slope.

4. Consider a plane wave spacetime with metric

$$ds^2 = -dx^+ dx^- - x_\perp^2 (dx^+)^2 + dx_\perp^2$$

Show that the Nambu-Goto string action can be solved completely in the lightcone gauge $x^+ = \tau$ in terms of some massive classical free field theory for $x_\perp$.

To do this, you should establish the following facts:

(a) The equation of motion of x^- can be written as an equation of motion of a massless free field (which is going to be x^+ in a background metric g^*)

(b) Like in class, choose a lightcone coordinate system $\sigma^\pm$ on the worldsheet.

(c) Show that one can solve the equation of motion of X^+ by choosing a lightcone gauge. Namely

$$X^+ = C(\sigma^+ + \sigma^-)$$

(d) Show that in the lightcone gauge the directions $X_\perp$ can be described by free massive scalar fields with respect to a flat metric on the worldsheet.

(e) Show that the equation of motion of X^+ lets you solve for X^- directly in terms of the other degrees of freedom (Solve the Virasoro constraints).

5. **Born-infeld brane**

Consider an acton for an extended p-brane (this means p space-dimensions plus one tome) of the form

$$S = -T_p \int d^{p+1}\sigma \sqrt{-\det(g^*)} \tag{5}$$

where g^* is the pull-back of the flat metric of Minkowsky space onto the worldvolume of the brane

$$g^*_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \tag{6}$$

(a) Show that the equations of motion of the p-brane can be written as follows

$$\partial_\alpha(\sqrt{-g^*}(g^*)^{\alpha\beta}\partial_\beta x^\mu) = 0 \tag{7}$$

(b) Show that a flat p-brane solves the equations of motion (this is, show that $X^{p+1} = \ldots = X^{D-1} = 0$ is a solution of the equations of motion. To do this, it is convenient to choose static gauge $\sigma^0 = X^0$, $\sigma^i = X^i$, for $i = 1, \ldots p$.

(c) By doing small fluctuations around the solution above, show that the transverse fluctuations propagate at the speed of light. Also show that T_p can be interpreted both as the mas per unit volume *and* the tension of the Dp-brane.