Week 3

Reading material from the books

- *Zwiebach, Chapter 12, 13, 21*
- *Polchinski, Chapter 1*
- *Becker, Becker, Schwartz, Chapter 2*
- *Green, Schwartz, Witten, chapter 2*

1 Polyakov action

We have found already that the variables $X^\mu(\sigma, \tau)$ are harmonic (solutions to the wave equation) with respect to the "trivial" metric defined on the worldsheet.

It is natural therefore to consider an action of the following form:

$$S \sim \int \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\mu \tag{1}$$

Where γ is an internal metric on the worldsheet. The form of the action above guarantees that the action is reparametrization invariant (diffeomorphism invariant in the math literature)

Clearly the equations of motion obtained from this action for the X^μ will be the same as for the Nambu-Goto string if the internal metric is equivalent to the induced metric.

Indeed, the idea is to treat the $\gamma_{\alpha\beta}$ as a dynamical variable and extremize the action with respect to γ. This is, we couple a free field theory to worldsheet gravity.

The variation of $\sqrt{-\gamma}$ is equal to $2^{-1}\sqrt{-\gamma}tr(\gamma^{-1}\delta\gamma)$ in matrix notation.

Similarly, the variation of γ^{-1} is proportional to $-\gamma^{-1}\delta\gamma\gamma^{-1}$ in matrix notation.

Doing this, we find that

$$\sqrt{-\gamma}(\gamma^{\alpha\delta} g_{\delta\epsilon,ind}\gamma^{\epsilon\beta}) = \sqrt{-\gamma}(g_{\delta\epsilon,ind}\gamma^{\delta\epsilon})\gamma^{\alpha\beta} \tag{2}$$

Lowering the indices we find that the worldsheet metric $\gamma_{\alpha\beta}$ differs from the induced metric by some scale factor f as follows

$$\gamma_{\alpha\beta} = f(\sigma) g_{\alpha\beta,ind} \tag{3}$$

So we can substitute this form of the metric in the Polyakov action. It is easy to see that no matter what the factor of f is, one recovers the Nambu-Goto action.

This means that the Polyakov action has more symmetry than the Nambu-Goto string action.

It's not just reparametrization invariant, but it also allows for the internal metric to be rescaled arbitrarily. This is, the Polyakov action has scale invariance on the string worldsheet.

This extra symmetry is called conformal invariance. It keeps angles fixed on the worldsheet, but changes lengths. We also call it Weyl invariance.

The fact that we get this property is natural. the extra metric adds three Lagrange multiplier degrees of freedom locally. However, reparametrization invariance has only redundancy by two functions (these define the coordinate system). This means we are adding too many degrees of freedom. Conformal (scale) invariance implies that one of these degrees of freedom decouples from the dynamics, leaving the correct number of degrees of freedom.

As seen before, we found that there always exists a coordinate system of lightcone coordinates in 2-D where the metric takes the following simple form

$$\gamma_{\alpha\beta} d\sigma^\alpha d\sigma^\beta \sim ds^2 = f(\sigma^\pm) d\sigma^+ d\sigma^- \tag{4}$$

By the Weyl rescaling property of the action, we can always choose γ so that $f = 1$ and the metric is constant.

We call this gauge the conformal gauge.

Equivalently, we are choosing a flat metric

$$ds^2 = -dt^2 + d\sigma^2$$

In this gauge the wave equations are trivially solved.

However, this gauge is not unique. The choice of t and σ are still affected by reparametrizations of σ^+ and σ^-. These are uniquely fixed in the light-cone gauge adapted to the conformal gauge, where we force the spacetime coordinate to have a very simple solution of it's equation of motion:

$$x^+ = At$$

However, in the end we have to supplement the equations of motion of the x^μ to make sure that the equations of motion of $\gamma_{\alpha\beta}$ are satisfied.

These equations read as follows

$$T^{\alpha\beta} = 0 \tag{5}$$

The stress energy tensor of the field theory vanishes.

These equations generalize the statement that in the theory of the point particle the Hamiltonian vanishes.

One could also imagine that the theory could have a cosmological constant and the Einstein-Hilbert action term on the worldsheet. The usual Einstein-Hilbert term in the gravity action turns out to be locally a total derivative. Therefore, it does not contribute to the equations of motion on the worldsheet.

The cosmological constant term is not Weyl-invariant and in fact, as written above it would lead to an inconsistency, because the scale factor of the metric would be drive either to infinite or to zero volume. - The scale factor mode is also called the Liouville mode. Conformal invariance implies that this mode decouples from the worldsheet dynamics (the dynamics of the rest of the fields is independent of rescalings).

It is easy to show that for the conformal field theory above (free fields) the stress energy component T^{+-} vanishes automatically. This is indeed true for any CFT. If we look at general solutions of the field theory, then it is not obvious that T^{++} and T^{--} vanish anymore. Indeed, they end up encoding the total energy of the system.

Energy momentum conservation then reads as the following two equations

$$\partial_+ T^{++} = \partial_+ T_{--} = 0 \tag{6}$$

$$\partial_- T^{--} = \partial_- T_{++} = 0 \tag{7}$$

Raising and lowering indices is done with an off-diagonal metric. That is why an upper $+$ index becomes a lower $-$ index, etc. This tells us that T_{--} is a function of σ^- alone. Similarly, T_{++} is a function of σ^+ alone.

From here, one finds the following conservation laws that are true for any solutions of the wave equations. Define the currents

$$T_f^+ = f(\sigma^-) T_{--} \tag{8}$$

$$T_f^- = 0 \tag{9}$$

It is easy to show that $\partial_\alpha T_f^\alpha = 0$ for any f. This means that in any conformal field theory there are always an infinite number of (left) conserved quantities.

One can do a similar construction with T_{++} that defines an infinite number of (right) conserved quantities.

The stress tensor constraints are the requirement that on solutions of the string equations of motion, we have $T_{++} = T_{--} = 0$. This is not automatic from the field theory. Indeed, it would seem naively that the set of solutions with this property is empty or trivial (the vacuum). However, the fact that the kinetic term for x^0 has the wrong sign means that we can lower the energy by "exciting" wave packets in x^0. This compensates for the fact that we are fixing the stress energy tensor to vanish: it is not a sum of squares and one can have cancellations between different terms.

The constraints $T_{++} = T_{--} = 0$ are also called Virasoro constraints.

- *In string theory we usually work on an analytic continuation of the Lorentzian metric for calculational purposes (we analytically continue $t \to it$). This is similar to the Wick rotation used in usual Feynman diagrams. In Euclidean coordinates, the $x^\pm$ become complex coordinates $z, \bar{z}$.*

  Current conservation for the stress energy tensor then implies that T_{zz} is holomorphic, and $T_{\bar{z}\bar{z}}$ is antiholomorphic. These holomorphic properties make it possible to use the tools of complex analytic functions to "solve" string theory problems and conformal field theory problems.

  In particular, the holomorphic properties of two dimensional surfaces make the study of Riemann surfaces as complex manifolds very important.

2 The classical closed string

We take a worldsheet with the topology $S^1 \times R$: the infinite cylinder.

The periodic variable is spacelike. The time-like variable can not be periodic because it would imply that spacetime (target space) has closed time-like curves.

In conformal gauge we can rescale the coordinate σ so that σ is periodic with period 2π.

The most general solution of the wave equation is then of the form

$$X^\mu(\sigma, t) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{10}$$

Including the periodicity of $\sigma \to \sigma + 2\pi$ we find that the most general solution is of the following form

$$X^\mu(\sigma, t) = a^\mu + p^\mu t + \sum_{n \neq 0} \frac{1}{\sqrt{n}} a_n^\mu \exp(in\sigma^-) + \frac{1}{\sqrt{n}} \tilde{a}_n^\mu \exp(in\sigma^+) \tag{11}$$

remember that $t \sim \sigma^+ + \sigma^-$. Reality of x^μ tells us that a_n and a_{-n} are complex conjugates of each other. This is also true for the $\tilde{a}_{\pm n}$. This form results from integrating the periodic functions $\partial_- x^\mu \sim \partial_- x_R^\mu$ and $\partial_+ x_L^\mu$ that have ordinary Fourier tranforms in σ. This is a mode expansion of the string.

From our analysis, the total momentum is given by contour integrals of $\partial_\alpha X^\mu$.

It is easy to show that the total momentum of the string is measured by p^μ. Similarly, a^μ controls the center of mass motion of the string: these are naturally canonically conjugate to each other.

The Virasoro constraints take the following form

$$T_{++} = 0 = \partial_+ x^\mu \partial_+ x^\mu \tag{12}$$

$$T_{--} = 0 = \partial_- x^\mu \partial_- x^\mu \tag{13}$$

Taking the constant term in the Fourier expansion of T_{++} we find that

$$(p^\mu)^2 + 4 \sum_{n>0} n \eta_{\mu\mu} |a_n^\mu|^2 = 0 \tag{14}$$

and similarly for T_{--}. This is, we can identify the mass squared of the string as a sum over modes.

In the quantum theory, the constant mode of the stress energy tensor is the Hamiltonian, and the $a_{\pm n}$ become raising and lowering operators. There are normal ordering ambiguities in the above equation, but up to a constant term, the sum over modes is the usual Hamiltonian in field theory for the left moving modes.

The above equation tells us that the mass spectrum of the string is a discrete set of integers in some units that are governed by the string tension. (This determines the so called slope of regge-trajectories)

We get the same mass squared for the right-movers. The total energy of the oscillators is an integer. This integer characterizing the energy levels is called the level of the string.

The statement that the "left moving mass" is equal to the "right-moving mass" is called the level matching constraint.

3 The classical open string

Let us consider the following worldsheet geometry, $[0, \pi] \times R$ where we have an open segment of a string.

The analysis is almost identical to the case of closed strings. However, we need to be more careful with the equations of motion of the x^μ.

The reason for this is that the action has the form

$$S \sim \int_0^\pi d\sigma \frac{1}{2} (\partial x^\mu)^2 \tag{15}$$

When we do variations of x^μ, we usually integrate by parts:

$$\delta S \sim \int_0^\pi \partial x^\mu \partial \delta x^\mu \sim -\int_0^\pi (\partial^2 x^\mu) \delta x^\mu \tag{16}$$

but the exact statement is as follows:

$$\delta S = -\int_0^\pi (\partial^2 x^\mu) \delta x^\mu + \int_0^\pi \partial(\partial x^\mu \delta x^\mu) \tag{17}$$

This generates a boundary term in the variation of the action by integrating a total derivative:

$$\delta S_B = \int dt \partial_\sigma x^\mu \delta x^\mu|_\pi - \int dt \partial_\sigma x^\mu \delta x^\mu|_0 \tag{18}$$

There are two natural ways to make this work. We choose variations where ∂x^μ vanishes exactly at the boundary for coordinate μ, this is, we have a Neumann or free boundary condition, or we choose restricted variations such that δx^μ vanishes at the boundary –Dirichlet boundary conditions. We can do this independently for each coordinate μ (or rotations of these coordinates).

The Dirichlet boundary conditions require that the end-points of the string be at fixed position X^μ. They can be different for the left and right ends of the string. These special positions in space break translation invariance.

This fixed position determines a special hyperplane in spacetime where strings can end. These hyperplanes (or generalizations) are topological defects in spacetime (objects). These objects have a name in string theory: D(irichlet)-branes.

The only case which leads to a Lorentz invariant theory in D dimensions is by choosing Neumann boundary conditions for all coordinates. It is conventional in most applications to choose the time direction to have Neumann boundary conditions. The other case are sometimes called S-branes. The S comes from space-like.

Furthermore, the left-moving and right-moving degrees of freedom of the string are connected via the boundary conditions of the string. This means that there are no independent left and right moving modes of the string any longer.

4 Lightcone quantization

We will begin now our discussion of quantization with the open string.

The idea is to set up our conformal gauge adapted to the lightcone, where the worlsheet metric is given by

$$ds^2 = -dt^2 + d\sigma^2 \tag{19}$$

and $\sigma \in [0, \pi]$. The choice of π is convenient for Fourier transforms on a segment. The spacetime coordinate is parametrized as follows $x^+ = P^+ t$. In these conventions the Hamiltonian is the canonically conjugate variable to t

$$-i\partial_t \sim H \sim -\frac{i}{P^+}\partial_{x^+} \tag{20}$$

and this is related to the spacetime energy-momentum component p_+

The action of the string is given by

$$S \sim \int_0^\pi d\sigma dt \frac{1}{2}[(\partial_t x_\perp)^2 - (\partial_\sigma x_\perp^2) + \int \frac{1}{2} p^+ \partial_t x^- \tag{21}$$

since p^+ is a parameter (dynamical variable), we see that it is conjugate to the zero mode of x^- (expanding x^- in fourier modes).

All other modes of x^- are determined from the Virasoro constraints. These are canonically conjugate to the Fourier modes of $x^=$ that are set to zero in lightcone gauge. Therefore they are redundant variables and are not quantized independently.

This means that apart from the zero modes of $x^\pm$, we only need to worry about $x_\perp$, that has $D-2$ field theory components.

The rest is just to work the canonical quantization of $x_\perp$.

It is easy to see that if we expand $x_\perp$ in normal modes (we are considering Neumann boundary conditions) then

$$x_\perp = \sum_{n=0}^{\infty} x_n(t)\cos(n\sigma) \tag{22}$$

We get an expansion in $\cos(n\sigma)$ because these are the modes that satisfy

$$\partial_\sigma \cos(n\sigma)|_{0,\pi} = 0$$

Plugging this expansion and noticing that

$$\int d\sigma \cos(n\sigma)\cos(m\sigma) = \frac{\pi}{2}\delta_{n,m}$$

for $n \neq 0$, we find that the action for each component of $x_\perp$ looks as follows for the modes $n \neq 0$

$$S \sim \sum_n \frac{1}{2}\dot{x}_n^2 - \frac{1}{2}n^2 x_n^2 \tag{23}$$

where the constant of proportionality is not important. We recognize this formula as the action for an infinite number of decoupled harmonic oscillators labeled by the integer n. The $n-th$ oscillator has frequency $\omega_n \sim n$.

We know from quantum mechanics that the hamiltonian for each of these modes can be written in terms of raising and lowering operators $a_n^\dagger$, a_n. We will use conventions where

$$[a_n^\dagger, a_m] = -\delta_{n,m} \tag{24}$$

and the Hamiltonian for each component of $x_\perp$ looks as follows

$$H = \sum_{n>0} n(a_n^\dagger a_n + \frac{1}{2}) \tag{25}$$

where we are keeping the usual zero point energy of the ordinary harmonic oscillators.

Since we are in a theory where the total energy matters (it affects the Virasoro constraint equations), we can not simply ignore the ground state energy and we are forced to confront the usual problem of infinities in quantum field theory.

IN string theory, one often writes

$$\sum_{n>0} n = -\frac{1}{12} \tag{26}$$

this result can be obtain by so-called zeta function regularization as follows. Define

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s} \tag{27}$$

This is an analytic function of s that is well defined (convergent) for $\Re(s) > 1$.

The function $\zeta(s)$ has a pole at $s = 1$ and can be analytically continued on the complex plane to define a meromorphic function of s. This analytic continuation satisfies

$$\zeta(-1) = -\frac{1}{12} \tag{28}$$

However, this hides all the physics.

A more correct derivation proceeds by understanding how the infinity in the expression looks like and applying the renormalization prescription (substraction by counterterms order by order in perturbation theory).

The first step is to regulate the expression. We need to be aware that if we change the interval length of the string to L, the energies are given by $n\pi/L$, instead of n. Now we choose a fixed energy to cutoff the sum over modes, but we want to cut off the modes smoothly as a function of L.

This is easy to do with an exponential function, so we regulate our zero point energy sum as follows

$$\sum n \to \frac{\pi}{L} \sum_n n \exp(-\epsilon n \pi/L) \tag{29}$$

Now we define $x = \exp(\epsilon\pi/L)$, and $\tilde{L} = L/\pi$ to rewrite the sum we get in terms of

$$\sum_n n x^n = \sum_n x\partial_x x^n = x\partial_x(\sum x^n) = x\partial_x(x/(1-x)) \tag{30}$$

This generating function is equal to

$$\frac{x}{(1-x)} + \frac{x^2}{(1-x)^2} = \frac{x}{(1-x)^2} \tag{31}$$

Now we want to take $\epsilon \to 0$ and we need to expand x as a power series in ϵ.

$$x = 1 - \epsilon/\tilde{L} + \frac{1}{2}\epsilon^2/(\tilde{L})^2 - \frac{1}{6}\epsilon^3/(\tilde{L})^3 \tag{32}$$

and we see that the formula has a double pole at $\epsilon = 0$ that is given by

$$\frac{(\tilde{L})^{-1}}{(\epsilon/\tilde{L})^2} \simeq \frac{\tilde{L}}{\epsilon^2} \tag{33}$$

Since this form of the energy is proportional to L, it is a constant energy density that can be cancelled if we add to the lagrangian density a constant term. This is a local counterterm to the action and it is a usual procedure in quantum field theory. This is generated to one loop order (order $\hbar$ with respect to the classical result). This constant term is like a cosmological constant for the worldsheet action and it is called the bare quantity (these are usually infinite). As we have seen before, the cosmological constant has to be zero by Weyl invariance. This refers to the total cosmological constant

$$\Lambda_{total} = \Lambda_{bare} + \Lambda_{1-loop} = 0 \tag{34}$$

This is the renormalization procedure. There is still a finite term in the expansion that leads to a physical energy (the Cassimir energy) by expanding the equation in ϵ.

$$E_{total} = \frac{\tilde{L}}{\epsilon^2}\frac{1 - \epsilon/\tilde{L} + \epsilon^2/\tilde{L}^2}{(1 - \epsilon/2\tilde{L} + \epsilon^2/6\tilde{L}^2)^2} - \tilde{L}/\epsilon^2 \tag{35}$$

The last term where we substract is the counterterm to the energy density (the bare energy density).

It's easy to see that the term of order $1/\epsilon$ cancels on it's own, and after some algebra one finds the desired result.

From here, apart from the zero mode quatization, the ground state energy of the string oscillators is

$$-\frac{D-2}{24} \tag{36}$$

For the closed string the result above doubles. One uses a periodic variable $\sigma \in [0, 2\pi]$ and one has double the degrees of freedom (one left moving oscillator and one right moving oscillator) with the same energies as above. Therefore one finds that the sum over the oscillator modes is just the double of the open string sum.

Now we need to understand how we use this to determine the spectrum of particle states of the open string.

To do this, we now need to quantize very carefully the zero modes.

The action for the zero modes of $x_\perp$ is given by

$$S = \frac{1}{2\pi\alpha'} \int dt \int_0^\pi d\sigma \frac{1}{2}\dot{x}_0^2 = \frac{1}{4\alpha'} \int dt \dot{x}_{\perp,0}^2 \tag{37}$$

Notice that this can be compared to a non-relativistic free particle of mass $\frac{1}{2\alpha'}$, so that the associated Hamiltonian is given by

$$\alpha' p_{\perp,0}^2 \tag{38}$$

Remember that $p_\perp$ generates translations in $x_\perp$, so in the quantum theory this is identified with the spacetime momentum along $x_\perp$. Since the zero modes act just like the degrees of freedom of the relativistic point particle, we know that they combine to give the "relativistic" combination that is associated with the mass squared of the particles.

We find that for the open string the Virasoro constraint associated to the total worldsheet energy being zero is

$$\alpha'(-m^2) + \sum_n n a_n^\dagger a_n - \frac{D-2}{24} = 0 \tag{39}$$

or equivalently

$$m^2 = \frac{1}{\alpha'}\left[\sum_n n a_n^\dagger a_n - \frac{D-2}{24}\right] \tag{40}$$

The combination

$$\sum_n a_n^\dagger a_n$$

is called N, the "level" of the string state. It takes integer values in states that diagonalize it, and it includes only the transverse coordinates to the lightcone. It also commutes with the zero mode degrees of freedom.

The wave functions can be written in terms of a Fock space of states multiplied by zero mode wave functions. Because of translation invariance the zero mode wave functions take the form (at fixed worldsheet time)

$$\exp(ik_\perp x_\perp + ik_- x^-) \tag{41}$$

We define a gound state at wave number k as

$$|k_-, k_\perp\rangle = |0\rangle \exp(ik_\perp x_\perp + ik_- x^-) \tag{42}$$

and we generate new states by acting with the oscillator raising operators on $|0\rangle$.

Thus the complete Hilbert space of a single string is characterized by the occupation numbers of each of the modes we have found, together with $k_\perp$ and k_-. The lightcone energy is obtained from the Virasoro constraint (generalized Wheeler-DeWitt equation) from the other quantum numbers. It is not an independent quantum number.

So apart from the zero point energy of the string, the spectrum of masses is given by integers.

One can also clearly see that there is one state (at zero momentum) for level 0. This is exactly the number of degrees of freedom of a scalar particle. We will now use the following convention:

$$a_n^\dagger = a_{-n} = (a_n)^\dagger \tag{43}$$

There are $D-2$ states at level one, characterized by the $(D-2)$ states

$$a_{-1}^i|0\rangle \tag{44}$$

These transform as a vector under the rotations of the D-2 transverse coordinates.

If one counts the degrees of freedom, one gets the same number of polarizations as those of a massless vector in D dimensions. A massive vector particle would have $D-1$ polarizations (in gauge theories, the extra polarization is the longitudinal degree of freedom of the Higgs mechanism that is eaten by the gauge boson).

In order for the above result to be compatible with Lorentz invariance, it must be the case that

$$-\frac{D-2}{24} + 1 = 0 \tag{45}$$

This is, $D = 24 + 2 = 26$.

The open bosonic string theory tells us that the space time dimension must be 26

At level 2 we get the following list of states

$$a_{-1}^i a_{-1}^j|0\rangle, a_{-2}^i|0\rangle \tag{46}$$

We get at first sight a generic two-index tensor of $SO(D-2)$, as well as a vector of $SO(D-2)$, for a total of $(D-2)^2 + D - 2$ components. However, the a_{-1} commute amongst themselves, so the tensor we get is symmetric in it's indices.

The little group of a massive particle in Lorentz unitary irreducible representations is

$$SO(D-1)$$

Given an unitary irreducible representation of

$$SO(D-1)$$

the method of induced representations of Wigner produces a unique irreducible unitary representation of the Lorentz group. Thus to check for Lorentz invariance one needs to show that one has all the states of such an irrep.

A traceless two index symmetric tensor of $SO(D-1)$ decomposes into a symmetric tensor and a vector under $SO(D-2)$. This is exactly what we found. At level two we therefore have a unique irreducible representation of the Lorentz group consisting of a massive traceless symmetric 2-tensor.

Finally, we can build a state of the maximum possible angular momentum at a given level by building a tensor with as many indices as possible. This can be done if we use only -1 oscillators.

The total angular momentum of such a state will be characterized at level n by a tensor with n vector indices, i.e., it will have spin n, and mass

$$m^2 = \frac{1}{\alpha'}(n-1) = \frac{1}{\alpha'}(J-1)$$

equivalently, we write this as

$$J = \alpha' m^2 + 1 \tag{47}$$

The constant α' that is derived from the string tension is the slope of this straight line. This slope is called the Regge slope, of the Regge trajectory. This type of equation fits qualitatively well with the spectrum of mesons and baryon states in QCD, so long as $\alpha' \sim 1GeV^{-1}$

4.1 The closed string spectrum

As we have argued before, the closed string has double as many oscillators as the open string.

In the open string, in the end we have only on set of Virasoro constraints. This is because the left moving degrees of freedom are related to the right-moving degrees of freedom by the boundary conditions. For the closed string we get two sets of Virasoro constraints. One for the left movers and one for the right-movers independently.

After this is done, the oscillator components of x^- are eliminated completely, but we still have the zero mode piece. To understand the Virasoro constraint for the zero mode piece we need to consider the total energy and total momentum of the string (these are determined by the constant modes of the Fourier expansion of the worldsheet stress energy tensor).

Again, the total Hamilotnian is

$$
\begin{aligned}
H_{tot} &= \sum_{left} n a^i_{-n} a_n + \sum_{right} n \tilde{a}^i_{-n} \tilde{a}^i_{-n} + \text{ zero modes} && (48)\\
&= N_L + N_R - 2(D-2)/24 + \text{zero modes} && (49)\\
&= 0 && (50)
\end{aligned}
$$

and

$$
P_{tot} = N_L - N_R = 0 \tag{51}
$$

This last statement reflects the fact that the vacuum is translationally invariant on the string, so that when $N_L = N_R = 0$ we get $P_{tot} = 0$. The second equation tells us that the left level and the right level must be the same value. This is called the "level matching constraint".

As we did before, we can look at the transverse components of the zero modes, and use those to understand the spectrum of masses.

The difference in this case is that we have now to integrate on the world-volume from zero to 2π. So the "effective mass" for the free particle associated to $x_{\perp,0}$ is now $\frac{1}{\alpha'}$ instead of $\frac{1}{2\alpha'}$. Thus the zero mode piece contribution to H is going to be

$$
H_{zeromode} \sim \frac{1}{2}\alpha' p_\perp^2 \tag{52}
$$

Again, at level 1 we get that $N_L = N_R = 1$, and we have a full 2-tensor of $SO(D-2)$ withno symmetries. This splits into an antisymmmetric 2-tensor, a symmetric 2 -tensor and a scalar of $SO(D-2)$. Again, compatibility with Lorentz invariance forces these states to be massless, as if it were otherwise one shows that one does not have complete unitary irreps of $SO(D-1)$.

This again produces the constraint $D = 26$.

Moreover, a massless symmetric 2-tensor can only arise from a gauge field associated to invariance under infinitesimal local translations (diffeomorphism invariance in spacetime). This is Gravity, and the associated quantum field is called a graviton.

The scalar field is called the dilaton. The antisymmetric tensor potential associated to the 2-tensor antisymmetric massless polarizations is called $B_{\mu\nu}$, the Kalb-Ramond field.

The zero mode contribution to the Hamiltonian reads as follows

$$\frac{\alpha'}{2} p_\perp^2 \tag{53}$$

From here we find that the masses satisfy in 26 dimensions

$$m^2 = \frac{2}{\alpha'}(N_L + N_R - 2) \tag{54}$$

Again, at level n, the mass squared is $(4n - 4)(\alpha')^{-1}$, and the maximum angular momentum one can have is $2n = J$. (One has as much angular momentum from the left and right movers).

Putting it all together we find that

$$J = (\alpha'/2)m^2 + 2 \tag{55}$$

The closed string tachyon is twice as heavy as the open string tachyon. Indeed, the level n closed string state is always twice as heavy as the corresponding open string state.

We also find that the open string Regge slpoe is twice as large as the closed string Regge slope.