Week 15

Reading material

- *Polchinski, chapter 10, chapter 12*
- *Becker, Becker, Schwartz, Chapter 4*
- *Green, Schwartz, Witten, chapter 4,5*

1 GSO projection

We have seen so far that it is possible to introduce spacetime fermions by using the R-NS formulation of the superstring: we introduce free fermions on the worldsheet.

We have also seen that after we bosonize the fermions, we can describe the corresponding (left-moing) vertex operators by

$$V \sim \Xi_\alpha \sim \exp(i\sum_j \pm\frac{H_j}{2})\exp(-\phi/2) = \exp(i\sum \frac{\epsilon_j}{2}H_j)\exp(-\phi/2) \quad (1)$$

And we can notate the α indices by the combinations of
pm that appear in the exponential. We also know that $\psi^\mu \sim \exp(\pm iH_j)$ for some j.

There are a total of 2^5 choices of signs, giving that a Dirac spinor in ten dimensions has 32 components.

If we consider a spacetime fermion, the left mover piece will be a spinor, and the right mover piece should be an integer spin object. The reason for this is that the total spin should be half-integer. If we consider a massless particle, then the right mover piece will be roughly $(\bar\partial X + k\psi\psi)\exp(ikX)$, and apart from the exponential piece, the conformal field has OPE's with similar conformal operators that go like integer powers of $(\bar z - \bar w)^{-1}$.

If we consider V_{+++++} and V_{-----}, we find that their OPE is given by

$$V_{+++++}(z)V_{-----}(w) \sim \frac{1}{(z-w)^{3/2}}\exp(-\phi) \quad (2)$$

Remember that the V have conformal dimension one, and that $\exp(-\phi)$ has conformal dimension equal to one half.

This power of $(z-w)^1/2$ indicates that the OPE between these vertex operators is not single valued (it changes sign as we take it around the other vertex operator). Indeed, if we integrate over the position of the second vertex operator we would get zero and this suggests that something is not working right: no scattering would be possible.

How do we solve this problem? The obvious answer is to say that one of the two vertex operators is not an allowed vertex operator. All vertex operators that are physical should give rise to single valued integrals: one should consider states that are mutually local on the worldsheet.

This also says that if we allow V_{+++++}, then we can not allow the tachyon vertex operator, as they would not be mutually local with respect to each other.

However, if we are allowing vertex operators of the form $\partial X + k\psi\psi$, this means that bilinears in the fermions are allowed. They operate on $\Xi_{\pm\pm\pm\pm\pm}$ by changing the sign of two of the entries: the bilinears can change a $\pm 12H_j \to \mp 12H_j$ for two different j.

This flipping of two signs keeps the product of the signs fixed.

Define the chirality of Ξ_{ϵ_j} by $\prod_j \epsilon_j$. We find that Ξ_{+++++} and Ξ_{-----} have opposite chirality. If we project onto only one chirality, we actually get a consistent set of OPE's. For example

$$V_{+++++}V_{----+} \sim \frac{1}{(z-w)} \exp(iH_5)\exp(-\phi) \sim \psi^\mu \exp(-\phi)$$

which does give rise to a mutually local set of operators. Moreover, these give rise to the ordinary vector particle quantum numbers of the right hand side that is the picture minus one version of a graviton vertex operator, so two spacetime spinors can combine into integer spin objects.

This projection in the NS sector keeps objects with two worldsheet fermions, but not with one (these are the $\psi\psi$ parts of the vertex operator.) Thus, the projection eliminates terms with an odd number of fermions. in particular, the tachyon vertex operator is eliminated by this procedure!

this projection is called the GSO projection, named after Gliozzi, Scherk and Olive.

Another thing we can find quickly is that the momentum on the worldsheet is measured by the operator

$$P^\mu \sim \oint \frac{dz}{2\pi i}(-i\partial X^\mu) \tag{3}$$

this is the left-moving part of a graviton vertex operator evaluated at zero momentum. In some sense, this is natural. We expect that a gauge particle couples to matter via a lagrangian of the form

$$j_I^\mu A_\mu^I \tag{4}$$

where I is a label of the symmetry we are gauging and j_I^μ is the conserved current. It can also have spacetime indices. At zero momentum for A, we measure the charge. This is, the graviton couples to momentum (the charge for the stress energy tensor).

One can check that this contour integral is BRST invariant: it changes physical states into physical states.

Similarly, we can consider a charge for the spinor, by taking a vertex operator for a spacetime fermion at zero momentum and just taking the left moving part.

These objects would give rise to a worldsheet current

$$Q_\alpha \sim \oint \frac{dz}{2\pi i} V_\alpha \tag{5}$$

This is also BRST invariant (it is after all a physical state at zero momentum).

A quick calculation shows that

$$P\{Q_\alpha, Q_\beta\} \sim \Gamma^\mu_{\alpha\beta} P_\mu \tag{6}$$

where we have included a picture changing operator to get the right hand side in picture zero.

This algebra is the algebra of supersymmetry. We find this way that the superstring has spacetime supersymmetry! Thus, the system we have has spacetime supersymmetry from the left movers. A similar property holds for the right-movers, so in the end this construction leads to a theory where translations and supersymmetries are gauged: a supergravity theory.

The choice of chirality is random, but one can distinguish the chiralities of the leftmovers relative to the right movers. If the two have opposite chiralities, the theory is called type IIA string theory(the two indicates thatwe have two supersymmetries). If the chiralities are the same, the theory is chiral and there are potential anomalies that might make it inconsistent. This theory is called type IIB string theory. In the limit where we only keep very low mass and energy excitations, we do not have enough energy to produce heavy string states.

Such a low energy limit would consist of only the graviton and it's partners under supersymmetry, and it leads to type IIA and type IIB supergravity.

A lot of effort in string theory is spent on trying to solve for solutions of the corresponding supergravity theories.

2 Cardy's formula

We are now in a position to exploit the concept of modular invariance. The end result we want to get i called Cardy's formula: it is a count of the asymptotic number of states in a conformal field theory. The maim idea is simple. As we have seen, the Euclidean partition function for a conformal field theory on a torus also encodes the list of states of a Lorentzian quantum field theory defined on a circle times time. This is also the list of local operators on the Euclidean worldsheet. Thus, the partition function contains all the information about the spectrum of operators of the theory, as well as their scaling dimensions.

This formula is given by

$$Z(q) = \sum \deg(n) q^n \tag{7}$$

where $\deg(n)$ is the degeneracy of the quantum field theory at energy n. Usually in this expression q is taken to be small for convergence. We can recover the degeneracy of states by doing a contour integral

$$\deg(n) = \oint \frac{dq}{2\pi i q} Z(q) q^{-n} \tag{8}$$

The integral is independent of the exact shape of the contour so long as the contour encircles zero one.

If we substitute $q = \exp(2\pi i \tau)$, the contour is over the periodic variable τ over a circle that encircles the period.

$$\deg(n) = \oint d\tau Z(\tau) \exp(-2\pi i n \tau) \tag{9}$$

we recognize that this is some Fourier transform of Z.

A contour near $q \sim 0$ is a contour where $\exp(-2\pi \Im m(\tau)) \sim 0$, so it is located at large $\Im m(\tau)$. We can slide the contour on the τ cylinder so that we are working near $\Im m(\tau)$ small. From the point of view of statistical mechanics, large $\Im m(\tau)$ is the low temperature region, while regions of small $\Im m(\tau)$ are the high temperature regions.

If we think that we are calculating the degeneracy of a gas at high energy, then we should consider the gas at high temperature. Thus, sliding the contour to the right temperature might facilitate the approximate computation of the degeneracy of states. This degeneracy is the entropy on a microcanonical ensemble, but to leading order, the microcanonical and the canonical ensembles give the same entropy.

This sliding of the contour has not helped us much so far. This is because $Z(q)$ near $|q| \sim 1$ is complicated, all terms in the series contribute.

Now, let us make the assumption of modular invariance. This is, let us assume that the partition function $Z(q)$ is modular invariant. In this case, we have that

$$Z(\tau) = Z\left(\frac{-1}{\tau}\right) \tag{10}$$

This is a natural assumption on a condensed matter system: if the theory is scale invariant, how we choose to look at a basis for a torus coordinate system should not affect the experiments. However, when we think of the region of small imaginary part of τ, under the modular transformation it can get sent to the regions of large imaginary value of τ, where the partition function is dominated by the low temperature regime, and thus only the first few terms contribute.

Now, let us also make the assumption of unitarity of the Eulcidean conformal field theory. This is called reflection positivity in Euclidean physics, and it is realized in many physical systems.

This means that the only representations of Virasoro that can be allowed are those that are unitary. IN particular, they are characterized by having positive conformal weight

$$h_\phi \geq 0 \tag{11}$$

The weight zero is allowed for the trivial operator, the identity. This is trivial to measure. One can also show that the stress tensor is always a descendant of the identity operator. This should be a physical operator that we can measure, so the conformal block of the identity should appear as part of the list of operators that are allowed, and it should appear only once. All other operators will have higher conformal weight.

The partition function $Z(q)$ near $q \sim 0$ can therefore be written as

$$Z(-1/\tau) \sim Z(q) \sim q^{-c/24} + \cdots = \exp\left(\frac{2\pi i c}{24\tau}\right) + \ldots \tag{12}$$

where all the other terms are exponentially suppressed.

We conclude that to a good approximation

$$\deg(n) = \oint d\tau \exp\left(\frac{2\pi i c}{24\tau}\right) \exp(-2\pi i n \tau) \tag{13}$$

and this can be evaluated by a saddle point approximation: we find the value of τ that extremizes the exponent:

$$\partial_\tau \left(\frac{2\pi i c}{24\tau} - 2\pi i n \tau\right) = 0 \tag{14}$$

This gives us

$$\tau = i\sqrt{\frac{c}{24n}} \tag{15}$$

We find then that

$$S(n) = \log(\deg(n)) \sim 2\pi\sqrt{\frac{cn}{6}} \tag{16}$$

by evaluating the exponent of the integral at the saddle. There are some subleading corrections that one can include. by doing a more rigorous analysis of the saddle point integral. This is called Cardy's formula.

The important observation, is that for large n, this covers the entropy. Notice that this depends on very little information at all. The central charge is the only quantity governing the problem.

The energy is $E \sim n$, and the interval of the base of the torus is taken to be equal to one. In the $1+1$ quantum field theory (the circle Hilbert space), we find the equation of state

$$S = 2\pi\sqrt{\frac{cE}{6}} \tag{17}$$

where S and E are the entropy density and the energy density (the interval is of size one).

We can use thermodynamics relations to calculate the temperature

$$T = \frac{\partial E}{\partial S} = \frac{6}{2\pi}\sqrt{E/6c} \tag{18}$$

We find the relation $E \sim T^2$, that the energy density scales like the temperature squared. This could have been guessed by dimensional analysis: in a relativistic field theory energy density has units of inverse length squared. Since the only scale in the system is the temperature, the relation can not be anything else. The central charge controls the coefficient of the proportionality relation.

Now, when we apply this to the relativistic string, either the bosonic or the fermionic, the degeneracy at level N would be governed by Cardy's formula, where the appropriate value of c is the central charge of the transverse lightcone degrees of freedom (not $c = 0$ which is the central charge of the full theory: after all, the ghosts have a non-unitary CFT and this would spoil some of the assumptions in the derivation of Cardy's formula: one just uses the modular invariance of the partition function that we have already calculated instead). These values are $c = 24$ for the bosonic string, and $c = 8+8/2$ for the fermionic string.

Also, the level is the mass squared of the particle, by the Virasoro constraints, so we find that the number of single particle states at mass m scales exponentially with m. This is called the Hagedorn behavior.

As a final application: we can consider a minimal model, with $c < 1$. In that case, unitarity tells us that the asymptotic growth of states is governed by c.

However, if we consider a general primary without null descendants, one finds that it has the same degeneracy of states as a free boson with $c = 1$.

The reason is that one can map the Virasoro energies $L_{-n_1} \dots L_{-n_k}|\phi\rangle$ to the counting of states in $a_{-n_1} \dots a_{-n_k}|0\rangle$, since both involve partitions of $n = n_1 + \cdots + n_k$. For a free boson, we know that $c = 1$, so this governs the degeneracy of states.

This gives us an alternative proof that the only unitary representations of minimal models belong to the series with null descendants, because this is the only way to cut sufficiently the degrees of freedom to be compatible with Cardy's formula.

Cardy's formula implies Hagedorn behavior.

This behavior essentially states that

$$\frac{dn}{dm} \simeq \exp \alpha m \tag{19}$$

Here we are using the fact that the level $N \simeq m^2$, so the density of states at mass m is the number of states at level $N \simeq m^2$. Since the number of states goes like $\exp(S)$, and $S \simeq \alpha\sqrt{N} \simeq \alpha m$, we get the result that we wanted. The Cardy formula appropriate for this discussion is the one that counts string polarizations in the lightcone. The full string theory that includes both the ghosts and the time-like x_0 is not unitary. The partition function is still modular invariant.

With such a growth of states, there is a "maximum" temperature, called the Hagedorn temperature. This corresponds to a phase transition in string theory. The canonical partition function for a gas of strings will will be as follows

$$Z[T] \simeq \sum_m \exp(-\beta m) \simeq \int dm \frac{dn}{dm} \exp(-\beta m) \tag{20}$$

So the integral diverges when $\alpha > \beta$. This means there is minimum β, and since $\beta \simeq 1/T$, there is a maximum T. The divergence is strong enough that the expectation value of the energy density blows up at this scale. Thus, not normal physics reservoir can take you past the Hagedorn temperature. One needs to be careful because the Hagedorn behavior assumes a gas of free strings.

At high energies one can expect that interactions will cutoff the number of states. In confining theories, which also are expected to look stringy at low energies, the specific heat at high temperature goes like N^2, where N is the number of colors. Planar diagram counting suggests that the string coupling constant here goes like $g_s \simeq 1/N$, so the specific heat goes like $1/g_s^2$. This is non-perturbative (not a Taylor series in g_s).