Week 5

Reading material from the books

- *Polchinski, Chapter 2, 15*
- *Becker, Becker, Schwartz, Chapter 3*
- *Ginspargs lectures, Chapters 3, 4*

1 Unitary representations of the Virasoro algebra

Now that we have the Virasoro algebra, given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{1}$$

we have to remember that the L_m are just the Fourier transform coefficients of T_{++} on the Lorentzian worldsheet.

By definition, the energy and momentum density are observables. Thus they are represented by a hermitian operator. In this way, we find that that T_{++} is observable. Since

$$T_{++} \sim \sum_{n=-\infty}^{\infty} L_n \exp(in(\sigma + \tau)) \tag{2}$$

we find that $L_0 = L_0^\dagger$, and that $(L_n)^\dagger = L_{-n}$.

A representation of an algebra on a space of matrices (that can be infinite) is called unitary, if for every non-zero vector in the representation we have that $||v\rangle|^2 > 0$. For this, we need an inner product, and an action of the adjoint operation on the algebra. If the inner product is positive definite, then the inner product is realized on a Hilbert space and can describe a quantum mechanical system.

Luckily, we have exactly that by having the hermiticity properties of the Virasoro generators.

Let us find the consequences of that statement. We will use L_0 as our Hamiltonian, as it is self-adjoint. Thus, we can diagonalize L_0 on our representation, and we can consider states such that $L_0|h, \alpha\rangle = h|h, \alpha\rangle$, where the$\alpha$ parametrize the degeneracy.

It is easy to see that L_{-n} raises the energy by n units. Given an eigenvector $|h,\alpha\rangle$, we can consider the state $L_{-n}|h,\alpha\rangle$, and we find

$$L_0 L_{-n}|h,\alpha\rangle = [L_0, L_{-n}]|h,\alpha\rangle + L_{-n}L_0|h,\alpha\rangle = nL_{-n}|h,\alpha\rangle + hL_{-n}|h,\alpha\rangle \quad (3)$$

Similarly L_n lowers the energy by n units.

For most field theories of interest, the spectrum of states has a minimum energy (the vacuum). Thus, we can not lower the energy indefinitely, and there must be states that have lowest energy in a representation. These states are called *primary*. They are characterized by the following relations

$$L_n|h\rangle = 0, \ \forall \ n > 0 \quad (4)$$

and also

$$L_0|h\rangle = h|h\rangle \quad (5)$$

Because L_0 is hermitian, it must be the case that h is a real number.

Via the commutation relations, we can show that all other states that can be generated from the algebra action on $|h\rangle$ are of the form

$$\sum_{n_i} a_{n_1 \ldots n_k} L_{-n_1} \ldots L_{-n_k}|h\rangle \quad (6)$$

This is because lowering operators can be moved all the way to the right, where they annihilate the state. Similarly, L_0 can be moved to the right all the way, where it gets replaced by h. Finally, we can use the commutation relations to order the L_n so that $n_1 \geq n_2 \geq n_3 \cdots \geq n_k$. This is, the n are ordered, and can be considered equivalent to a partition of $N = n_1 + \cdots + n_k$ into k integers. The number of partitions of N is $P(N)$. By definition, $P(0) = 1$. And for example, $P(1) = 1$, $P(2) = 2$ (these are $\{1,1\}$ and $\{2\}$). Also, $P(3) = 3$ ($\{3\}, \{2,1\}, \{1,1,1\}$. Then we have $P(4) = 5$.

The integer N is called the *level of a state*. It has energy $h + N$.

States that are of this form, for $N \geq 1$ are called *descendants*.

Clearly, there are only finitely many states at a given level: there are only finitely many partitions of N.

We want to determine that the representation is unitary. Thus, we want to require that any state at level N has positive norm.

At level zero, we have $||h\rangle|^2 = 1$, by choice of normalization. At level one, the only state we have is

$$L_{-1}|h\rangle \quad (7)$$

using the unitarity relation, we find that

$$\langle h|L_1 L_{-1}|h\rangle = 2h\langle h|h\rangle \tag{8}$$

so we find that $h \geq 0$. If we have $|L_{-1}|h\rangle|^2 = 0$, we should in the end conclude that $L_{-1}|h\rangle = 0$. Indeed, if we have this one state at level one, it will be unique, and it will be orthogonal to every other state in the representation of the Virasoro algebra. States that are orthogonal to everything else (including themselves) are called null states. This distinguishes them from vectors that have zero norm in a space with an indefinite inner product.

Similarly, if we consider $L_{-m}|h\rangle$ for very large m, we find that

$$|L_{-m}|h\rangle|^2 = \left(\frac{c}{12}(m^3 - m) + 2mh\right)\langle h|h\rangle \tag{9}$$

this is positive definite for very large m only if $c \geq 0$.

Thus we have found that for the Virasoro algebra representation to be unitary, it must be the case that $h \geq 0$ and $c \geq 0$. Actually, we can do better than this. At level 2, we find two states

$$|h,1,1\rangle = L_{-1}^2|h\rangle, |h,2\rangle = L_{-2}|h\rangle \tag{10}$$

We want every single linear combination of these two states to be positive definite. Thus we need that

$$|a_{1,1}|h,1,1\rangle + a_2|h,2\rangle|^2 \geq 0 \tag{11}$$

this gives us

$$|a_{1,1}|^2\|keth,1,1|^2 + |a_2|^2||h,2\rangle|^2 + a_{1,1}^* a_2\langle h,1,1||h,2\rangle + a_{1,1}a_2^*\langle h,2||h,1,1\rangle \geq 0 \tag{12}$$

we see that this is a general quadratic form acting on the two dimensional complex vector a_{11}, a_2, given by the 2×2 matrix

$$\begin{pmatrix} \langle h|L_2 L_{-2}|h\rangle & \langle h|L_1^2 L_{-2}|h\rangle \\ \langle h|L_2 L_{-1}^2|h\rangle & \langle h|L_1^2 L_{-1}^2|h\rangle \end{pmatrix} = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix} \tag{13}$$

In order for this matrix to define a positive definite inner product, it must be the case that $\det(M) \geq 0$, and the same is true for all the minors. But if $h, c \geq 0$, the minors are trivially positive.

The null states are going to be combination that are annihilated by this matrix (this will be called the Kac matrix).

The determinant is given by

$$2(16h^3 - 10h^2 + 2h^2c + hc) = 32(h - h_{1,1}(c))(h - h_{1,2}(c))(h - h_{2,1}(c)) \quad (14)$$

where $h_{1,1}(0) = 0$, and $h_{2,1}$ and $h_{1,2}$ are given by

$$h_{1,2}, h_{2,1} = \frac{1}{16}(5 - c) \mp \frac{1}{16}\sqrt{(1 - c)(25 - c)} \quad (15)$$

If the determinant vanishes, we have at least one null state. In this case this can happen if h is equal to one of the roots of the polynomial.

The roots are all real if $c < 1$ or $c > 25$. If $c > 25$, the roots $h_{2,1}$ and $h_{1,2}$ are both negative, so it is not an issue. Also, if the roots are complex, there is no real zero and hence no possibility of a null state in the set of representations we are considering.

Furthermore, we recognize that $h = 0$ is a root of the similar determinant (of a 1×1 matrix at level 1). This will be a general feature: if one has a null state at level N, $|h + N\rangle$, there will be many more null states that descend from it in the Kac formula, ass there will be many zero eigenvalues.

Generally, if we have a null state $|n\rangle$, then any linear combinations of $|n\rangle$ acted on by Virasoro generatorswill be null. This is proved as follows:

$$\langle a|n\rangle = 0, \quad \forall a \quad (16)$$

Therefore, if we consider a general combination

$$\mathcal{O}|n\rangle \quad (17)$$

where $\mathcal{O}$ is a polynomial in the Virasoro generators, we conclude that

$$\langle a|(|\mathcal{O}\rangle|n\rangle) = (\mathcal{O}^\dagger|a\rangle)^\dagger|n\rangle = \langle b|n\rangle \quad (18)$$

where $b = \mathcal{O}^\dagger|a\rangle$. The right hand side is trivially zero.

Thus the set of null states is closed under the action of Virasoro (in the language of algebras, this is a called a submodule). We can define a smaller representation of Virasoro by taking all null states and making them identically to zero. Such type of representation is called a *Verma module*.

Since the representations of Virasoro that we are considering have positive energy, the null submodule will also have this property (a state with minimum

energy) and it is of the same type as before, with a highest weight null state (a state that is both a primary and a descendant). If $|n\rangle$ is the first state in a representation that is null, then $L_k|n\rangle = 0$ for all $k > 0$, as we will reduce the energy, and there are no null states before n. This shows that it is a primary state.

The number of elements of the given null submodule at level N is $P(N)$. If $|n\rangle$ is a descendant at level n relative to h, then at level M there will be $P(M-n)$ null descendants of $|n\rangle 0$. All of these are null eigenvectors.

So far, we find that there are special null states at $h = h_{2,1}$, $h = h_{1,2}$ that are null for values of $0 < c \leq 1$ that give rise to possible null states. If $c > 1$, the product is positive definite for all $h \geq 0$, so there are no possible null states.

We also find that if h is in between $h_{2,1}$ and $h_{1,2}$ or if $h < 0$, then there is a negative norm state at level 2. But if $h < 0$, there is also a negative normed state at level one. Thus, we are excluding further intervals.

We can generalize this to arbitrary level. One will have a really big quadratic form to analyze. Fortunately, most of the information is contained in the determinant of the quadratic form.

This factorizes into various roots for h. This formula is known as the Kac determinant formula, and it is a very long and hard computation. One finds that

$$\det(M_N(c,h)) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)} \tag{19}$$

where $P(N-pq)$ is the number of partitions of $N-pq$ into integers.

It is useful to introduce the number m, given by

$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \tag{20}$$

which is a different parametrizaton of c.

The numbers $h_{p,q}$ are then given by

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \tag{21}$$

One also verifies easily that

$$c = 1 - \frac{6}{m(m+1)} \tag{22}$$

One sees easily that in the determinant formula each root is repeated exactly as dictated by partitions of $N-pq$, so it counts all the null descendants of a highest weight null state at level pq.

Obviously if the Kac determinant is negative, there are negative normed states. We also can see that the boundary between unitary and non-unitary representations should be a change of sign in the Kac determinant. This happens (potentially) when the determinant vanishes.

New zeros can be isolated.

It is also known that the determinant vanishes at various values of h at $c = 1$, but it does not change sign.

A full analysis is beyond what we will do. However, one can show that for $c > 1$ there are no negative normed states, whereas for $c < 1$ there are generically negative normed states, unless two different hp, q take the same value. This happens when

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \ldots \tag{23}$$

and then the only possible values of h are given precisely by the $h_{p,q}$, where p, q are in the range

$$m - 1 \geq p \geq q \geq 1 \tag{24}$$

Such models are very constrained, and they have a name. They are called minimal models.

From the list of states, the state with $h = h_{p,q}$ has a null descendant at level pq. This can be calculated explicitly.

2 The operator-state correspondence

A very important technical aspect of conformal field theories is the operator state correspondence.

This is based on the following observation: the metric of D-dimensional flat euclidean space can be written in spherical coordinates, and removing the point at the origin of the coordinate system, the metric is equivalent by a Weyl transformation to the metric of a cylinder, whose base is a sphere in

$D-1$ dimensions. This is as follows

$$ds^2 = dr^2 + r^2 d\Omega_{D-1}^2 \tag{25}$$

$$= r^2 \left[\left(\frac{dr}{r} \right)^2 + d\Omega_{D-1}^2 \right] \tag{26}$$

Now consider the variable $t = \ln(r)$. Then $dt = dr/r$, so the above metric (up to the factor $\exp(2t)$ can be written as

$$\exp(-2t) ds^2 = \left(dt^2 + d\omega_{D-1}^2 \right) \tag{27}$$

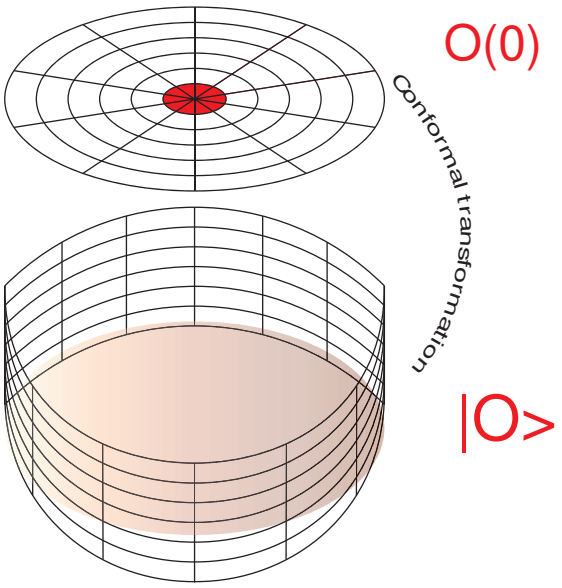
The metric on the right hand side is the metric of a sphere times a line. This is an infinite cylinder with a sphere as a base. In the case of two dimensions, the base is a circle, and the metric of the cylinder is also flat.

This rescaling is a global symmetry of conformal field theories. This means that the theory on the flat pane and the cylinder are mapped into each other. If we interpret t as time, notice that the infinite past with respect to t corresponds to the value $r = 0$.

By a similar token, t going to infinity corresponds to the infinite point in the complex plane.

Now assume that we insert an operator at the origin. $\mathcal{O}(0)$. This operator changes the behavior of the quantum fields at the origin.

However, this origin is absent in the cylinder. To change the fields at $t \to -\infty$, we need to consider a quantum state that is different from the vacuum. *In the graphic below the letter in the ket is an $\mathcal{O}$, not a zero*



This means that to each local operator in the field theory, we can associate a state in the field theory. Similarly, if we cut the path integral at some value of t and declare that at that time t we have inserted some given state, we can evolve the state backwards in time until the corresponding circle shrinks to a point. This operation should be described by some information located at the origin, so we need to insert an operator to describe how the state differs from the vacuum near the origin.

This correspondence between states and operators acting on the vacuum is called the operator-state correspondence.

The operator associated to a state is also called vertex operator. We will study this dictionary more carefully in what follows.

Let us assume that we have a primary state $|h\rangle$. We should associate to this state an operator $\mathcal{O}_h(0)$. If $|h\rangle$ is primary, then by definitions the operator $\mathcal{O}_h(0)$ is a *primary field*, or a primary operator. A descendant of a state, will correspond to a descendant field (or operator).

The action of $L_n|h\rangle$ is given by a contour integral of T at time $t = 0$ on the cylinder. This becomes a contour integral of T around the corresponding operator insertion.

As we have already seen, contour integrals correspond to commutators, so our dictionary is

$$L_n|h\rangle \to [L_n, \mathcal{O}_h](0) \tag{28}$$

From this, we can understand what characterizes a primary field:

$$[L_n, \mathcal{O}_h](0) = 0, n > 0 \quad (29)$$
$$[L_0, \mathcal{O}_h](0) = h\mathcal{O}_h(0) \quad (30)$$

This will be encoded in the OPE of the operator with the stress tensor:

$$T(z)\mathcal{O}_h(0) \sim \sum_{n>0} z^{-n} A_n(0) \quad (31)$$

Doing the contour integrals associated to L_n, as given by

$$L_n \sim \oint \frac{z^{m+1}}{2\pi i} T(z) \quad (32)$$

we find that $A_n = 0, for n > 2$, and that

$$T(z)\mathcal{O}_h(0) \sim \frac{h}{z^2}\mathcal{O}_h(0) + O(1/z) \quad (33)$$

Also, notice that L_1 corresponds to contour integrals without any z factor. We can take the contour along some real slices instead, and do the "Minkowsky analytic continuation" to flat Lorentzian space, rather than the radial analytic continuation taking us to the cylinder.

We discover that $\int T$ is related to the total energy, or momentum operator in flat space, so this generates ordinary translations of the fields.

Thus, $[L_1, \mathcal{O}(0)] = \partial\mathcal{O}(0)$. We find this way the complete singular terms for the OPE for primary fields

$$T(z)\mathcal{O}_h(0) \sim \frac{h}{z^2}\mathcal{O}_h(0) + \frac{1}{z}\partial\mathcal{O}_h(0) \quad (34)$$

We can prove that a primary field transforms as follows under general conformal transformations

$$\mathcal{O}_h(z)dz^h \to \mathcal{O}_h(\xi)d\xi^h \quad (35)$$

for $\xi = \xi(z)$. This is,

$$\mathcal{O}_h(\xi) = \mathcal{O}_h(z)\left(\frac{d\xi}{dz}\right)^{-h} \quad (36)$$

Similarly for right movers.

The quantity $h - \bar{h}$ is the spin: the phase acquired under rotations of the operator. The quantity $h + \bar{h}$ is the dimension: the eigenvalue under dilatations.

We will show later that the complete set of operator correlation functions of all fields can be completely determined by the correlation functions of the primaries. This will be due to a Ward identity involving the stress tensor and the other fields.

Some last things to remember:

- Unitarity it not a natural concept in Euclidean field theory. Thus, there are many critical points in condensed matter systems that correspond to non-unitary CFT's.

- In string theory we end up in Euclidean field theory because it is easy to do complex analysis and contour integrals. The real physical string happens in Lorentzian signature, and it is possible that there are some hidden caveats that would let us define more versions of string theory that do not admit the Euclidean continuation...

- **Very important!** If the central charge does not vanish $c \neq 0$, then the stress tensor is *not a primary field*. This means that there is an anomalous transformation for the stress tensor under coordinate changes, and setting $T = 0$ is not a covariant statement.