Week 7

Reading material from the books

- *Polchinski, Chapter 7*

1 Modular Invariance

We are ready now to describe the property of Modular invariance.

The idea is to first define what we would call a character of the Virasoro algebra for a conformal block. This is a counting of how many states N_k at level $k$descend from a common primary $[\phi]$ (this is an invariant of the conformal block) and it is given by

$$\chi([\phi]) = \sum_k N_k q^{L_0 - c/24} = q^{h_\phi - c/24} \sum N_k q^k \tag{1}$$

For fields that have no null descendants we have $N_k = P(k)$, so we can write for general primaries

$$\chi([\phi]) = q^{h_\phi - c/24} \prod_{i>0} (1 - q^i)^{-1} \tag{2}$$

This sum is exactly like $\exp(-\beta H)$, where we identify $L_0 - c/24$ with the Hamiltonian, and so long as $q = \exp(-\beta)$. As such, it is a partition function.

The product form is an identity. The idea is that

$$\frac{1}{1 - q^s} = \sum_{k=0}^{\infty} q^{s \cdot k} \tag{3}$$

So that this counts one, when we have k copies of s. A general partition is a division into integers in decreasing order $N = s_1 + \dots s_k$ with $s_1 \geq s_2 \geq \dots \geq s_k$. We can express this in terms of how many copies of 1 there are, how many copies of 2 etc. So that $N = n_1 \cdot 1 + n_2 \cdot 2 + \dots$ and there is one partition for such a division. When we multiply the infinite series, we get a 1 for each copy of $n_1 \cdot 1 + \dots + n_m \cdot m$, this, we get 1 for each partition. If we fix the value of the sum, we get as many states at level N as there are partitions. We found that this was the degeneracy of the "typical" Virasoro UIR.

This is very natural in general. Usually a character of a group element g is a trace over some representation of g, $\chi(g)_R$. This can also be done for Lie algebras, where we take traces of Casimir elements. All of these are "gauge invariant" quantities.

For $SU(2)$ for example, any element of the group can be conjugated to $\exp(i\beta L_z)$ (rotations about the z axis). This function of β then counts how many states we have at each spin $z$value $\pm s$. The character defined above is essentially the same thing as we have done here, over an unitary (and in particular irreducible) representation of the Virasoro algebra.

When we consider a true conformal field theory, there is also the right moving part, for which we use

$$\chi([\phi]) = \sum_{k,k'} N_{k,k'} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \tag{4}$$

where we allow the possibility that $c_L \neq c_R$ by having both a c and $\bar{c}$. Again, one would expect that this is just a product

$$q^{h_\phi - c/24} \prod_{i>0} (1 - q^i)^{-1} \bar{q}^{\bar{h}_\phi - \bar{c}/24} \prod_{i>0} (1 - \bar{q}^i)^{-1} \tag{5}$$

So we find the same product function again.

Now we want to understand this a bit better: we consider the variables $q, \bar{q}$ to be complex numbers, but we will require that they are conjugates of each other $q^\dagger = \bar{q}$. We write

$$q = \exp(-\beta + i\theta) \tag{6}$$

and then $\bar{q} = \exp(-\beta - i\theta)$. Using this expression, we find that the partition function goes as

$$\exp(-\beta(L_0 + \bar{L}_0) + i\theta(L_0 - \bar{L}_0)) \tag{7}$$

so it is like a thermal partition function $\exp(-\beta H)$ that has been twisted by a rotation with angle θ (this is an imaginary chemical potential for $L_0 - \bar{L}_0$).

We can think that this results from doing a computation of a path integral on the cylinder with length β, and that has been rotated by angle θ. This is the usual notion that $\exp(-\beta H)$ can be written as a path integral.

Consider a flat torus geometry. This is represented by having an identification of the plane with $z \simeq z + 1$, and $z \simeq z + \tau$. The complex variableτ is called the modulus of the torus. It can be chosen so that $\Im m(\tau) \geq 0$ and

$-1/2 \leq \Re e(\tau) < 1/2$. Taking $\tau \to \tau + 1$ is like a rotation in θ by 2π. So that we want to take $i\theta - \beta = 2\pi i \tau$. With these conventions the real part of τ becomes the angle of rotation (up to a factor of 2π) and the imaginary part, which is positive, upon rotation by i becomes a negative number, so it makes the variable q to be such that $|q| < 1$. One can then show that the product formula above is convergent.

This is our first instance of a new special function. The Dedekind eta function defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{8}$$

We see that these characters are in general proportional to $\eta(\tau)^{-1}$, where we have that $q = \exp(2\pi i \tau)$.

Now, if we sum over all states, and not just the descendants of one representation, we get the partition function on the periodic torus without restrictions. Basically, we are now doing Conformal field theory on the torus. The torus partition function is then

$$Z(\tau, \bar\tau) = \mathrm{Tr}(q^{L_0 - c/24} \bar q^{\bar L_0 - \bar c/24}) \tag{9}$$

the shifts of $c/24$ and $\bar c/24$ arise from going from the plane to the cylinder and *including the Cassimir energy*. This arises from the Schwarzian derivative and defines the correct energy for the vacuum.

The basic idea of modular invariance is that if we rescale the size of the torus, nothing should change. Also nothing changes if we rotate the torus periodicity on the complex plane. We can arrange that the side parametrized by τ becomes the new side that corresponds to 1. We do this by multiplying all sides by $1/\tau$. However, the side that was one before now corresponds to $1/\tau$, but this has the imaginary part of the wrong sign. So, we take the negative of it to put it in the right domain of positive imaginary part.

This is represented in figure 1.

Thus, we should have that the partition function behaves simply under the change $\tau \to -\frac{1}{\tau}$, and also under $\tau \to \tau + 1$ (we shift the angle by an extra 2π). Indeed, it should essentially not change at all.

We would want to have

$$Z(\tau, \bar\tau) = Z(\tau + 1, \bar\tau + 1) = Z(\frac{-1}{\tau}, \frac{-1}{\bar\tau}) \tag{10}$$

We call this property *Modular Invariance*.

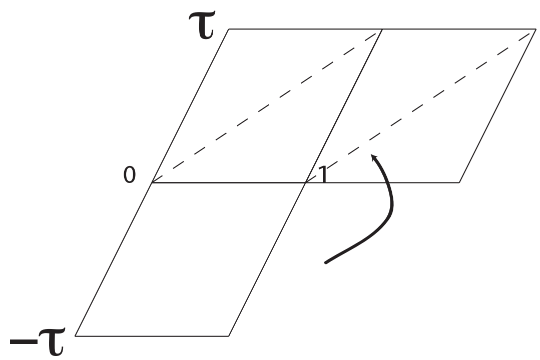

Figure 1: The tau transformations

The map $\tau \to \tau + 1$ is called the T operation. The map $\tau \to -\tau^{-1}$ is called the S operation. Together they are the generators of the modular group

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{11}$$

with $ad - bc = 1$ and a, b, c, d integers. These can be interpreted as a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{12}$$

The T operation is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{13}$$

The S operation is

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{14}$$

The modular fundamental domain is the region of τ from which every other possible τ can be obtained by an element of $SL(2, \mathbb{Z})$. This is depicted in figure (2).

An inversion $\tau- \to -\frac{1}{\tau}$ has a fixed circle of points where $\tau \simeq \exp(i\theta)$ sends points of the circle to itself. This means that we can take $|\tau| \geq 1$. Also, with $\tau \to \tau + 1$, we can choose τ to be in a strip of width one. This starts at $-1/2$ to $1/2$ by convention. To each of these corresponds a unique τ.

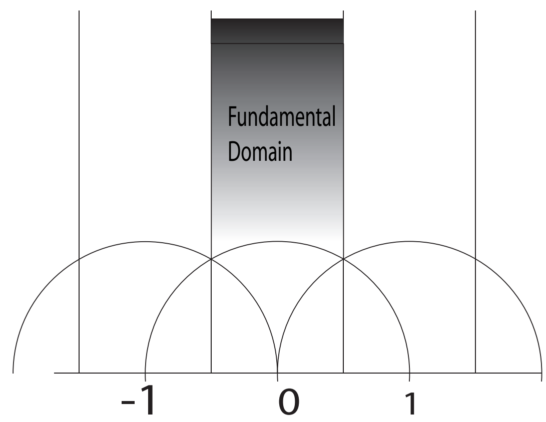

Figure 2: The fundamental domain

Sometimes we can allow something weaker than modular invariance, but we will get to that later.

What we need now are examples.

2 The free scalar field

Let us start with a free scalar field. With the mode decomposition in the cylinder, we have a mode per integer a_{-n}, for $n = 1, \ldots \infty$, of energy $L_0 = n$, and similarly $\tilde{a}_{-n}$ with energy $\bar{L}_0 = n$. We also have a zero mode shared between the left and right movers. The wave function for the zero mode needs to be of the form $\exp(ipx_0)$, and has an energy $L_0, \bar{L}_0 = p^2/2$.

This is the dimension of the operator $:\exp(ipX):$, and this is what we map the wave function $\exp(ipx_0)$ into using the operator state correspondence. The other modes are mapped as follows $a_{-n} \to \partial^n X$. It is easy to check that this gives us a 1-1 map of states in the Fock space (+zero mode) into local operators. This also preserves the energy (above the ground state) mapping to the dimension of the operator. Remember that the ground state energy shift is entirely accounted by the Schwarzian derivative.

We need to be careful in the partition function, because the parameter p is continuous. The non-zero mode part can be easily written. For each a_n we get a partition function of a boson $(1-q^n)^{-1}$ and we need to multiply them. Being careful with the central charge we find that the non-zero mode part is given by

$$Z(\tau,\bar{\tau}) = \eta(\tau)^{-1}\eta(-\bar{\tau})^{-1}Z_{0-mode} \tag{15}$$

Notice that here we get $-\bar{\tau}$ for the antiholomorphic one. This is the one that guarantees that the imaginary part is positive.

The Z_{0-mode} should address the fact that the spectrum of p is continuous. We should therefore write

$$Z_{0-mode} \simeq \int dp q^{p^2/2}\bar{q}^{p^2/2} = \int dp \exp(-2\pi\Im m(\tau)p^2) = \frac{A}{\sqrt{\Im m(\tau)}} \tag{16}$$

Let us check modular invariance. First under T η^{-1} picks a phase under $\tau \to \tau+1$ from the $q^{-1/24}$ (after all $q \to \exp(2\pi 1)q$ in the transformation), but this is cancelled by the corresponding phase in $\bar{q}$. So $\eta\bar{\eta}$ is manifestly invariant under $\tau \to \tau+1$. So is the integrand (the imaginary part of τ does not change).

$$\boxed{T : \surd}$$

More precisely, we get that

$$\eta(\tau+1) = \eta(\tau)\exp(2\pi i/24) = \exp(\pi i/12)\eta(\tau) \tag{17}$$

Now let us check S. First, let us understand how $\tau \to -1/\tau$ works. We have that $\tau = \Re e(\tau) + i\Im m(\tau)$, and that

$$-\frac{1}{\tau} = -\frac{\bar{\tau}}{\tau\bar{\tau}}$$

So that

$$\Im m(-\frac{1}{\tau}) = \frac{\Im m(\tau)}{\bar{\tau}\tau}$$

So from here we get that

$$Z_{0-mode} \to Z_{0-mode}\sqrt{\tau}\sqrt{\bar{\tau}}$$

Another property of the η function is that

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \tag{18}$$

Similarly

$$\eta(-1/(-\bar{\tau})) = (i\bar{\tau})^{1/2}\eta(-\bar{\tau}) \tag{19}$$

We see that the factors of i cancel in the product (the function is real for $\Re\tau = 0$). So we get that

$$Z(\tau,\bar{\tau}) \to Z_{0-mode}\sqrt{\tau}\sqrt{\bar{\tau}}(\tau\bar{\tau})^{-1/2}\eta^{-1}(\tau)\eta^{-1}(-\bar{\tau}) = Z(\tau,\bar{\tau}) \tag{20}$$

$$\boxed{S:\surd}$$

We have proven that the partition function of the free scalar is modular invariant.

2.1 Mutual locality

Two operators $\mathcal{O}_{12}(z,\bar{z})$ are said to be mutually local if their OPE is single valued.

This is, they are defined by

$$\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2) \simeq \sum_{\mathcal{O}_3} \frac{C_{ijk}}{(z_1-z_2)^{h_1+h_2-h_3}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}}\mathcal{O}_3(z_2,\bar{z}_2) \quad (21)$$

over all local operators of whatever weights are allowed.

This is single valued if $h_1+h_2-h_3-(\bar{h}_1+\bar{h}_2-\bar{h}_3)\in\mathbb{Z}$. This is usually a physical requirement that operator insertions do not depend on brach cuts in the complex plane.

In particular, if we take an operator and it's conjugate, we find that in their OPE the identity operator shows up, with dimensions $(h,\bar{h})=(0,0)$. Mutual locality in this case requires that $2(h_1-\bar{h}_1)\in\mathbb{Z}$.

Now let us consider the case of the character of the corresponding representation. All descendants differ in values of $h,\bar{h}$ by integers with respect to the primary. The character starts as

$$q^{-c/24}\bar{q}^{-\bar{c}/24}q^h q^{\bar{h}}\Big(\sum_{k,\tilde{k}\in\mathbb{N}} N_{k,\tilde{k}}q^k\bar{q}^{\tilde{k}}\Big) \quad (22)$$

where $N_{0,0}=1$. We want this to be invariant under the T modular transform. For this, we need to send $q\to\exp(2\pi i)q$, $\tau\to\tau+1$, and also $\bar{q}\to\exp(-2\pi i)\bar{q}$.

The phase of the character is $\exp(2\pi i(c-\bar{c})/24)\exp(2\pi i(h-\bar{h})$.

For unitary field theories with an identity operator character, $h=\bar{h}=0$ and we want this phase to vanish. We find then that $c-\bar{c}$ must be a multiple of 24. Assuming this, for any other allowed operator we find that

$$h-\bar{h}\in\mathbb{Z} \quad (23)$$

We thus find that modular invariance implies a constraint on the values of $(h,\bar{h})$ which is *stronger than mutual locality* of an operator and it's dual.

Indeed, modular invariance under T forces us to consider only collections of fields that are mutually local.

This is very important when considering CFT's like the $c = 1/2$ Ising model.

The reason is that there are only three possible conformal primaries of dimensions $0, 1/16, 1/2$. The need of modular invariance makes it so that we also need a $\bar{c} = 1/2$ on the right movers, unless we want to go to very high central charge. In this case, the left moving dimension and right moving dimension will need to be paired up.

3 The $m = 3$ minimal model

Before we go into the topic of modular invariance, it is convenient to study for a while the simplest minimal model so that one has an understanding of how the system works in general.

For $m = 3$, we find that the central charge is equal to

$$c = 1 - \frac{6}{m(m+1)} = 1 - \frac{1}{2} = \frac{1}{2} \tag{24}$$

The dimensions of the minimal fields are $h_{1,1} = 0$, $h_{2,1} = 1/2$, $h_{1,2} = 1/16$.

The only field of dimension zero should be the identity (this is for unitary conformal field theory). Thus the $h_{1,1}$ field represents the simplest conformal field.

Now, let us consider a field for $h_{2,1} = 1/2$. For convenience, let us call it ψ. At this stage we are only interested in left moving fields.

If we consider the following two point function, it is uniquely fixed by conformal symmetry and the weight of ψ, so we find that

$$\langle \psi(z)\psi(w) \rangle = \frac{1}{z - w} \tag{25}$$

Notice that if we exchange z and w the correlation changes sign. Thus the field ψ has Fermi statistics. It is tempting to consider ψ as a free field. If that is the case, then the action for ψ should be given by a lagrangian of conformal weight $(1, 1)$. For free fields, dimensions add up, and the only choice we have is to have a quadratic action in ψ with one $\bar{\partial}$ operator to make the right mover dimension.

We find that

$$\mathcal{L} \sim \int d^2 z \frac{1}{2} \psi \bar{\partial} \psi \tag{26}$$

We recognize this lagrangian as a version of the Dirac lagrangian for a chiral field in two dimensions. The idea is that we should consider

$$\bar{\psi} \gamma^\mu \partial_\mu \psi \tag{27}$$

where we use $\gamma^1 \sim \sigma^1$ and $\gamma^2 \sim \sigma^2$, and we ask that ψ is Majorana, so that$\bar{\psi} \sim C\psi$, where C is a charge conjugation matrix.

The two component spinor ends up splitting into two decoupled degrees of freedom, one left mover and one right mover. We can prove that we get the same action in the end.

The stress tensor for the left mover should have the right dimension. Also, since $\psi(z)$ is a fermion, we have that $\psi(z)^2 = 0$ (this is the Pauli exclusion principle). Thus, the most general form of the stress tensor that one could have by dimensional analysis is

$$T \sim a : \psi \partial \psi : +b : \psi^4 := a : \psi \partial \psi : \tag{28}$$

but the second term vanishes by the Fermi statistics.

Proper normalization of the OPE of T with itself (using Wicks theorem) shows that $a = \frac{1}{2}$ and we also find that $c = 1/2$ by direct computation.

Thus the $c = 1/2$ minimal model is just the free fermion theory.

If we consider a free fermion on the cylinder, we would imagine that there is an expansion in Fourier modes. Since ψ is a conformal primary, we find that if ψ is single values on the plane, then on the cylinder

$$\psi_{cyl}(w) = \left(\frac{\partial w}{\partial z} \right)^{-1/2} \psi(z) \tag{29}$$

and $w \sim \log(z)$, so

$$\frac{\partial w}{\partial z} \sim \frac{1}{z} \tag{30}$$

and we find that on the cylinder the fermi field should be anti-periodic (the square root is double valued, so one has a double cover of the cylinder on which the field is single valued, but once around the loop we need to change sign). The fact that on the cylinder the natural vacuum is the one that has anti-periodic boundary conditions is surprising.

We call this boundary condition the Neveu-Schwartz (NS) sector.

Similarly, one could consider periodic boundary conditions, which is called the Ramond (R) sector.

One naturally considers a Fourier mode expansion on the cylinder, just like we did for the stress tensor. Back in the plane, this is a Laurent series expansion. We set

$$\psi(z) = \sum \psi_n z^{-n-1/2} \tag{31}$$

The label n is a half integer for the NS sector, and an integer for the R sector. The shift by a half is the conformal weight of ψ and follows from equation 29

The coefficients ψ_n can be obtained by doing contour integrals of the ψ field itself.

$$\psi_n = \oint \frac{dz}{2\pi i} z^{n-1/2} \psi(z) \tag{32}$$

The powers of z guarantee that the integrand is single valued on a contour around the origin.

By a straightforward computation one can turn the OPE into commutation relations for the Fourier modes (except that for Fermions we should use an anticommutator) :

$$\begin{aligned}
\{\psi_n, \psi_m\} &= [\oint \frac{dz}{2\pi i} z^{n-1/2}, \oint \frac{dw}{2\pi i} w^{m-1/2}] \langle \psi(z)\psi(w)\rangle && (33)\\
&= [\oint \frac{dz}{2\pi i} z^{n-1/2}, \oint \frac{dw}{2\pi i} w^{n-1/2}] \frac{1}{z-w} && (34)\\
&= \delta_{n+m,0} && (35)
\end{aligned}$$

The z contour goes around the insertion of w, and then we are still left with the w contour. The z contour forces $z = w$, and then to get a non-zero answer we need that $m + n - 1 = -1$.

This was calculated using the periodic conditions on the plane, which makes the n, m labels half-integers (the labeling is done according to the cylinder convention).

One would expect the same anticommutation relations to hold for periodic boundary conditions on the cylinder (anti-periodic on the plane), except that now m, n are integer moded.

The OPE with the stress tensor can be used to show that

$$[L_0, \psi_n] \sim n\psi_n \tag{36}$$

so that the ψ_{-n} raise the energy by n units, and the ψ_n lower the energy by n units. This convention is the same that is used for the stress energy. Again, we find that the OPE's encode all the commutation relations in a nicely packaged form.

One should notice that there is a special case where $n = m = 0$, where we obtain that

$$\psi_0^2 = \frac{1}{2} \tag{37}$$

This particular mode does not change the energy. This is called a zero mode. If we have more than one flavor of fermion, the commutation relations in the R sector for the zero modes (let us say for d fermions) give us

$$\{\psi_0^i, \psi_0^j\} \sim \delta^{ij} \tag{38}$$

We find that this is equivalent to a Clifford algebra for d dimensions. Thus, the ground state in the R sector is a representation of this Clifford algebra. This is the definition of spinors of $SO(d)$.

Also, a straightforward calculation in the case of a single spinor shows that there is a null descendant at level 2.

$$L_{-1}^2 \psi = \partial^2 \psi \tag{39}$$

and $L_{-2}\psi$ should be proportional to $: \psi \partial \psi \psi : (0) + \partial^2 \psi(0) + : \psi^5 : (0)$, the most general object of the right dimension. However, Fermi statistics imposes that any term with repeated objects $\psi(0)^2$ should vanish, so this is proportional to $\partial^2 \psi$.

This shows that at level two there is a null state, as the level two vector space is degenerate.

We can also use the commutation relations in the R sector to reconstruct the fermion propagator for the anti-periodic conditions in the plane (these are antiperiodic conditions around the origin, so there is a branch cut from the origin to infinity, where there is a square root branch cut).

We can compute the Green s function for this case to find that

$$\langle \psi(z)\psi(w) \rangle_A \sim \frac{\frac{1}{2}\left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}\right)}{z - w} \tag{40}$$

Let us derive this using complex analysis, rather than things we know just from free fields.

Indeed, near the origin, we should have branch cut in z. Adding a ψ does not modify this branch cut. This branch cut indicates the insertion of a local operator at the origin, that we will call $\sigma(0)$. If we do an OPE around the origin, we find that

$$\psi(z)\sigma(0) \sim \frac{1}{z^{1/2}}\mu(0) \tag{41}$$

where σ and μ are the corresponding primary fields (they do not have to be the same, but they should have the same dimension, and each of them introduces a branch cut at the origin).

The field σ is called a twist field. The square root os z is the indication that we have a branch cut. Dimensional analysis tells us that It is equal ton the dimension of ψ. We also know that the most singular terms of OPE between primaries involve primary fields themselves, so if σ and ψ are primary, so should be μ.

We should also have a single pole at w, from the OPE coefficient with $\psi(w)$.

Since we have a branch cut that reaches infinity, we should also have an OPE coefficient in the dual coordinate system $z \to \xi = \frac{1}{z}$, so that near infinity it should behave like $\xi^{-1/2}$ in the new coordinate system. But remember that fields transform under coordinate transformations, so the asymptotics near infinity for the correlation should go like $z^{-1/2}$. If we put this together, we find that

$$\langle \psi(z)\psi(w)\rangle_A \sim \sqrt{z} f(z,w) \tag{42}$$

where $f(z,w)$ is single valued in z. It approaches zero when z is near infinity, and it has a pole at zero and at w. Such a function is completely characterized by its poles, zeros and asymptotics so that

$$f(z,w) \sim \frac{a}{z} + \frac{b}{z-w} \tag{43}$$

We can get b from the residue at $z = w$ of the OPE, which should be one. Therefore

$$Res_{z=w}\frac{z^{1/2}b}{z-w} = 1 \tag{44}$$

so $b = w^{-1/2}$. By using symmetry arguments (thinking of doing the same analysis in the variable w), we find that the answer could not be anything other than equation 40.

Also, since we know the form of the stress tensor $T(z) \sim \frac{1}{2} : \psi\partial\psi : (z)$, this is the same as

$$T(z) = \lim_{w \to z} \left(\frac{1}{2}\psi(w)\partial\psi(z) - \frac{1}{2(z-w)^2} \right) \tag{45}$$

We can in this way compute the correlator

$$\langle T(z) \rangle_A \tag{46}$$

We are interested in the double pole at $z \to 0$ that indicates the dimension of the primary field at zero. We find that

$$\langle T(z) \rangle_A \sim \frac{1}{16z^2} \tag{47}$$

so that $h_\sigma = \frac{1}{16}$. This is, the twist field is the other conformal primary associated to $h_{1,2} = \frac{1}{16}$.

It is also easy to show that the twist field has a null descendant at level 2. This is because on the cylinder, at level 2 we have the only possible states

$$\psi_{-2}|0\rangle_R + \psi_{-1}^2|0\rangle_R + \psi_0\psi_{-2}|0\rangle_R + \psi_0\psi_{-1}^2|0\rangle_R \tag{48}$$

but $\psi_{-1}^2 = 0$, and the ground state is degenerate $|0\rangle_R$ and $\psi_0|0\rangle_R$.

Thus, at level two we have the same degeneracy as the ground state (rather than a double degeneracy as expected in general). This shows that at level two there are null states with respect to Virasoro.

A more complete argument would consider that the stress tensor carries fermion number two, so only states with an even number of fermion excited states relative to the initial vacuum state can be accessed by acting with Virasoro operators.

Now let us look at the characters of the Virasoro algebra

For example, for the free fermion we have that for antiperiodic boundary conditions, we have both the identity and the free fermion conformal blocks, and

$$\chi(1) + \chi(\psi) = q^{-1/48} \prod_{i=0}^{\infty} (1 + q^{i+1/2}) \sim tr_A q^H \tag{49}$$

where we have to take the trace in the sector of anti-periodic boundary conditions.

This is just the sum over all modes of the partition function of the free fermions with anti-periodic boundary conditions. Characters are additive on the set of representations [1].

Also, notice that the χ_ψ corresponds to an odd number of fermions. These are the terms with half-odd energy. So the two characters can be separated using the fermion number $(-1)^F$

$$\chi(1) - \chi(\psi) = q^{-1/48} \prod_{i=0}^{\infty} (1 - q^{i+1/2}) \sim tr_A (-1)^F q^H \tag{50}$$

(Notice also that $(-1)^F$ commutes with T and in particular H)

Similarly we can analyze the system with periodic boundary conditions. Notice that for free fermions on a circle there usually is a zero mode. This mode satisfies $\{\psi_0, \psi_0\} = 1$, so in principle it could be realized simply by $\psi_0 = 1/\sqrt{2}$. However, we usually also want to have the fermion number $(-1)^F$ associated to parity $\psi \to -\psi$ being a symmetry of the vacuum. This can not happen for the above simple realization. Instead we need $(-1)^F$ and ψ to anti-commute. This defines a two dimensional Clifford algebra with $(-1)^F/\sqrt{2}$ being the second generator. Indeed, if we also have right moving fermions, and if they have a zero mode, we would get $\{\psi_0, \bar{\psi}_0\} = 0$, so $(-1)^F$ can be seen as $\bar{\psi}_0$, while $(-1)^{\bar{F}}$ can be identified as ψ_0 itself.

The smallest representation of this algebra is two dimensional. This way we define two states of opposite fermion parity:

$$|\frac{1}{16}\rangle_{\pm} \tag{51}$$

and to this, we have an associated character

$$\chi_- + \chi_+ = 2q^{1/16-1/48} \prod_{i=1}^{\infty} (1 + q^i) \tag{52}$$

while we can also distinguish the two by $(-1)^F$, with the obvious result that

$$-\chi_- + \chi_+ = 0 \tag{53}$$

To decompose the sums into the corresponding characters, we introduce projectors

$$P_\pm = (1 \pm (-1)^F)/2 \tag{54}$$

[1] In mathematics, they provide a functor that takes direct sums of representations into sums of functions of q. These are very similar to Chern classes

, so that

$$\chi(0) = tr_A \left(\frac{(1+(-1)^F)}{2} q^H \right) \tag{55}$$

$$\chi(\psi) = tr_A \left(\frac{(1-(-1)^F)}{2} q^H \right) \tag{56}$$

$$\chi_\pm = tr_P \left(\frac{(1\pm(-1)^F)}{2} q^H \right) \tag{57}$$

The subscripts indicate the type of boundary conditions on the spatial axis.

For fermions, the usual partition function with periodic boundary conditions in time measures

$$tr(-1)^F \exp(-\beta H)$$

while the "canonical" partition function is done with anti-periodic boundary conditions.

Again, we are seeing that the different partition functions we computed before, are interpretable in terms of path integrals with *anti-periodic or periodic* boundary conditions.

Since we have a two dimensional field theory and we are doing also a translation, we see that we have to do a path integral over a torus. The base of the torus is 2π.

The height of the torus is $2\pi\Im m(\tau)$, but we also need to twist the top by a translation in σ.

What we find this way, is that the characters we computed above can be written as fermions on a complex torus, with identifications $\zeta \to \zeta + 1$, and $\zeta \to \zeta + \tau$ up to rescaling from σ, t.

However, invariance under $\tau \to \tau + 1$ is a different choice for the fundamental domain of the torus in the covering space.

We can easily see that $tr_A q^H$ is equivalent to the path integral with AA conditions on the fermions. In the same way $tr_A q^H (-1)^F$ is equivalent to a path integral with AP conditions on the fermions. (First letter is space periodicity, second one is time periodicity)

It is easy to see that as we shift $\tau \to \tau + 1$, the AA and the AP get exchanged into each other. There is a simple invariant: $AA + AP$. $AA - AP$ transforms into minus itself. However, this is just the left moving sector. We can make two invariants as follows:

$$|AA + AP|^2, |AA - AP|^2 \tag{58}$$

This is the statement that the left and right moving operators must be paired somehow.

The operation of taking $\tau \rightarrow \tau + 1$ continuously is called a Dehn twist.

Similarly $|PA|^2$ and $|PP|^2$ are invariant under $\tau \rightarrow \tau + 1$.

Under these circumstances, it is easy to see that if we start with anti-periodic boundary conditions on both "space and time" and we do a Dehn twist and we follow it with an S operation, we end up with *periodic boundary conditions* on the space direction.

Finally, we should note that the final answer of the partition function forces us to consider the same periodicity for left and right-moving spinors, plus a projection of the type

$$tr \left(\frac{1 + (-1)^{F+\bar{F})}}{2} \right) q^H \bar{q}^{\bar{H}} \tag{59}$$

These types of sums over different "topological sectors" of fermions determined by the boundary conditions go by the name of GSO projection. For higher genus Riemann surfaces, different choices of ferrmion boundary conditions end up being related to periodic and anti-periodic functions on the lattice (although the fermions are not exactly periodic and anti-periodic themselves). Roughly, they transform like a square root of dz and there are choices on the signs we need to patch different coordinate sets.

These "periodicity conditions" are called spin-structures. They are classified in two classes: even and odd spin structures. They are determined by wether the left movers have and even or an odd number of zero modes. Similarly we have even and odd spin structures for the right-movers.

In the case of the $c = 1/2$ model, (which incidentally corresponds to the Ising model at criticality), the spin structures of left and right movers are the same.