# Fundamentals of Radiative Transfer

see Rybicki & Lightman ch. 1

- Energy <u>Flux</u> - amount of radiative energy passing through an element of area $dA$ in time $dt$

  $$dE = F\, dA\, dt$$

  $F$ has units $erg\ s^{-1}\ cm^{-2}$

  Note: Measure of energy carried by <u>all</u> <u>rays</u> passing through a given area.

- Definition of <u>Specific Intensity</u>

  More detailed description of radiation is to give the energy carried along by <u>individual</u> <u>rays</u>.

  Consider set of rays which differ infinitesimally from the given ray... passing through $dA$ with solid angle $d\Omega$ of given ray. Energy crossing $dA$ in $dt$ per $d\nu$ is

  $$dE = I_\nu\, dA\, dt\, d\Omega\, d\nu$$

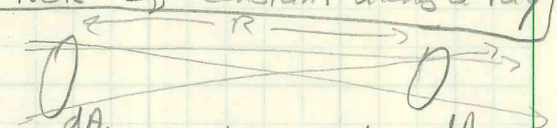
$dE_1 = I_{\nu_1} dA_1\, dt\, d\Omega_1\, d\nu_1 = dE_2 = I_{\nu 2} dA_2\, dt\, d\Omega_2\, d\nu_1$

$d\Omega_1 = dA_2 / R^2$

so $I_{\nu_1} = I_{\nu 2}$


$[I_\nu]$ $erg\ s^{-1}\ cm^{-2}\ ster^{-1}\ Hz^{-1}$

$\boxed{\text{Note } I_\nu = \text{constant along a ray}}$




$d\Omega_1$ = solid angle subtended by $dA_2$ at $dA_1$.

① Mean Intensity (zeroth moment of $I_\nu$)

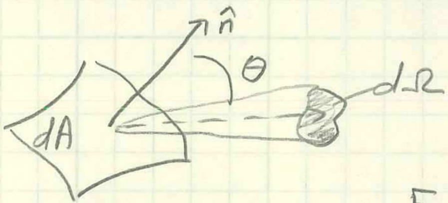
$$J_\nu = \frac{1}{4\pi} \int I_\nu\, d\Omega \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

Radiation Density $\quad u_\nu = \frac{4\pi}{c} J_\nu$

② Flux (first moment of $I_\nu$)

suppose we have rays in all directions, then the flux from $d\Omega$ is lowered by the effective area $\cos\theta\, dA$.

$I_\nu$ isotropic $\Rightarrow$ no net flux




$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$

③ Radiation Pressure (second moment of $I_\nu$)

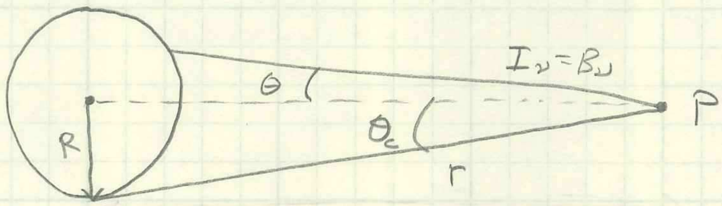
To get the flux of momentum normal to $dA$ recall that the momentum of a photon is $E/c$.

Momentum Flux along a ray is $\frac{1}{c}\frac{dE}{dA\,dt} \Rightarrow \frac{1}{c} dF_\nu$

Multiply by another $\cos\theta$ to get the component of momentum flux normal to $dA$.

$$P_\nu = \frac{1}{c}\int I_\nu \cos^2\theta \, d\Omega \qquad \text{dynes cm}^{-2}\ \text{Hz}^{-1}$$

- Flux from a Uniformly Bright Sphere




At P the specific intensity is $B_\nu$ if the ray intersects the source.

$$F = \int I \cos\theta \, d\Omega$$

$$= B \int_0^{2\pi} d\phi \int_0^{\theta_c} \cos\theta \sin\theta \, d\theta$$

$$= 2\pi B \int_0^{\theta_c} \sin\theta \, d(\sin\theta)$$

$$= 2\pi B \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\theta_c}$$

Note $\theta_c = \sin^{-1}\frac{R}{r}$

$$\boxed{F = \pi B \left(\frac{R}{r}\right)^2}$$

Brightness

Note: At stellar surface $R = r$, and $F = \pi B$.

## Emission and Absorption Coefficients

1) Emissivity

$$j_\nu = \frac{1}{4\pi} n_u A_{ul} h\nu \phi_\nu$$

↑ line profile describes the probability that the emitted photon will have frequency between $\nu$ and $\nu + d\nu$.

2) Attenuation Coefficient

describes the "net" absorption, i.e., true absorption minus stimulated emission.

$$K_\nu = n_l \sigma_{l\to u}(\nu) - n_u \sigma_{u\to l}(\nu)$$

$$= n_l \sigma_{l\to u} - \frac{n_u \sigma_{u\to l}}{n_l \sigma_{l\to u}} n_l \sigma_{l\to u}$$

$$= n_l \sigma_{l\to u}(\nu)\left[1 - \frac{n_u/n_l}{\sigma_{l\to u}/\sigma_{u\to l}}\right]$$

$$\frac{\sigma_{l\to u}}{\sigma_{u\to l}} = \frac{g_u}{g_l}$$

$$K_\nu = n_l \sigma_{l\to u}(\nu)\left[1 - \frac{n_u/n_l}{g_u/g_l}\right]$$

If the absorbers come into equilibrium with the radiation field, then the relative population of the levels depends on their energy difference, or equivalently, the photon energy.

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-h\nu/kT}, \text{ where } -h\nu = E_l - E_u$$

$$K_\nu = n_l \sigma_{l\to u}(\nu)\left[1 - e^{-h\nu/kT_{ex}}\right]$$

(a) Stimulated emission can often be neglected because the upper levels of atoms and ions usually have negligible populations.

(b) At radio and sub-mm frequencies, it is important to include stimulated emission.

# Radiative Transfer



$$dI_\nu = -\kappa I_\nu \, ds + j_\nu ds$$

"emissivity"

"attenuation coefficient"

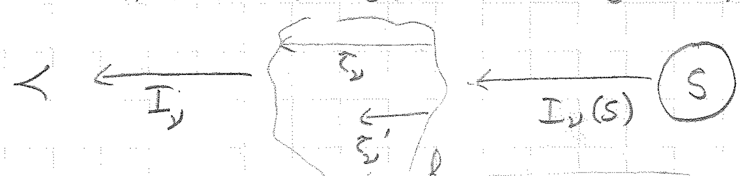
(1) Point source (star or quasar): Absorption is against a point source but emission is generally into the whole sphere. Fraction of emission towards the observer is negligible.

$$I_\nu^{-1} dI_\nu = -\kappa ds$$

$$I_\nu = I_\nu(0) e^{-\int \kappa ds}$$

$$I_\nu(\tau) = I_{\nu,0} e^{-\tau}$$

(2) Radio observations have large beam, emission from all the atoms in the beam cannot be ignored




Note: $\tau(x) = \int_s^{obs} \kappa ds$

$$dI_\nu + I_\nu d\tau = \frac{j_\nu}{\kappa_\nu} d\tau \qquad \text{Transfer Eqn.}$$

<u>Integrating Factor</u> $e^{\int_{obs}^{l} \kappa ds}$

$$e^{\int_{obs}^{l} \kappa ds} dI_\nu + e^{\int_{obs}^{l} \kappa ds} I_\nu d\tau = j/\kappa \, e^{\int_{obs}^{l} \kappa ds} d\tau$$

Note Limits

$$\int_s^{obs} d\left[e^{\int_{obs}^{l} \kappa ds} I_\nu\right] = \int_0^{\tau_\nu} j/\kappa \, e^{\int_{obs}^{l} \kappa ds} d\tau$$

$$e^{\int_{obs}^{obs} \kappa ds} I_\nu - e^{\int_{obs}^{s} \kappa ds} I_\nu(s) = \text{"}$$

$$I_\nu = e^{-\tau_\nu} I_\nu(s) + \int_0^{\tau_\nu} j/\kappa \, \exp\left[\int_{obs}^{s} \kappa ds - \int_l^s \kappa ds\right] d\tau'$$

$$\boxed{I_\nu = e^{-\tau_\nu} I_\nu(s) + \int_0^{\tau_\nu} j_\nu/\kappa_\nu \, e^{-[\tau_\nu - \tau_\nu']} d\tau'}$$

The source and its attenuation

... Each source in the cloud and the attenuation between that point in the cloud and the observer.

If cloud in LTE, then Kirchhoff's Law applies

$$j(\nu) = K(\nu)\, B_\nu(T),$$

i.e., emission and absorption are equal for a blackbody.

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\nu = I_\nu(s)\, e^{-\tau_\nu} + B_\nu(T) \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, d\tau'$$

Let $x = -(\tau_\nu - \tau_\nu') = \tau_\nu' - \tau_\nu$

$dx = d\tau_\nu'$

$$\int_{-\tau_\nu}^{0} e^x\, dx = e^0 - e^{-\tau_\nu}$$

$$I_\nu = I_\nu(s)\, e^{-\tau_\nu} + B_\nu(T)\left(1 - e^{-\tau_\nu}\right)$$

e.g., When cloud is very diffuse $\tau_\nu \approx 0$, and we get no emission from the cloud

$$B_\nu(T)\left(1 - e^{-\tau_\nu}\right) \longrightarrow B_\nu(T)\,(1-1) \approx 0$$

When cloud is very thick, $\tau_\nu \gg 1$, $B_\nu(T)(1-0) \rightarrow B_\nu(T)$

then cloud radiates as a BB.

And we do not see the source behind it

$$\lim_{\tau_\nu \to \infty} I_\nu(s)\, e^{-\tau_\nu} \Rightarrow 0$$

# Radio Astronomy

$$\frac{h\nu}{kT} \ll 1 \implies e^{h\nu/kT} - 1 \approx 1 + \frac{h\nu}{kT} - 1 + \cdots \approx \frac{h\nu}{kT}$$

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2}\,\frac{kT}{h\nu} \approx \frac{2\nu^2 kT}{c^2}$$ Rayleigh-Jeans Law

$B_\lambda = B_\nu \cdot \frac{c}{\lambda^2}$

$$I_\nu = I_\nu(s)\,e^{-\tau_\nu} + \frac{2\nu^2 kT}{c^2}\left(1 - e^{-\tau_\nu}\right)$$

Define antenna temperature $T_A \ni$

$$\frac{c^2 I_\nu}{2\nu^2 k} = I_\nu(s)\,\frac{c^2}{2\nu^2 k}\,e^{-\tau_\nu} + T\left(1 - e^{-\tau_\nu}\right)$$

$$\boxed{T_A = T_{A,s}\,e^{-\tau_\nu} + T\left(1 - e^{-\tau_\nu}\right)}$$

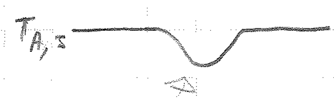
$T$ ← cloud Temperature

(a) $T_{A,s} > T \implies$ Absorption Line

(i) $\tau_\nu \ll 1$ at line center

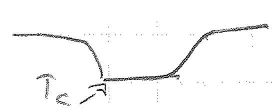
$$T_A = T_{A,s}(1 - \tau_\nu) + T\left(1 - (1 - \tau_\nu)\right)$$

$$= T_{A,s} - \underbrace{(T_{A,s} - T)\,\tau_\nu}$$




(ii) $\tau_\nu > 1$ at line center, the $e^{-\tau_\nu} \to 0$

and $T_A \approx T$ see cloud not the source at line center




but $T_A > T$ away from line center

Radio Astronomy

- <u>Brightness temperature</u> is defined to be the temperature such that a blackbody at that temperature would have specific intensity $B_\nu(T_{BB}) = I_\nu$.
- In LTE, the brightness temperature is the actual thermodynamic temperature.

$$B_\nu(T) \equiv \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

so $$T_B(\nu) \doteq \frac{h\nu}{k} \ln\left[\frac{2h\nu^3}{c^2 I_\nu} + 1\right]^{-1}$$

- $T_B$ is a non-linear function of temperature.
- Astronomers define <u>antenna temperature</u> so that it is linear in the intensity. At radio frequencies, $h\nu \ll kT$, and $B_\nu(T) \approx \frac{2h\nu^3}{c^2}\frac{kT}{h\nu}$.

Hence, $$T_A \equiv \frac{c^2}{2k\nu^2} I_\nu$$

## Radio Astronomy - Maser Lines

Upper levels are populated at radio and sub-mm frequencies.

(1) Include spontaneous emission

$$I_\nu = I_\nu(0)\,e^{-\tau_\nu} + \underline{B_\nu(T_{ex})\,(1-e^{-\tau_\nu})}$$

(2) Include stimulated emission

$$\kappa_\nu = n_\ell\,\sigma_{l\to u}\,(1-e^{-h\nu/kT_{ex}})$$

Collisional or radiative excitation of a level higher than u that then decays to populate level u causes a <u>population inversion</u> when the <u>pumping</u> is rapid compared to the processes that depopulate level u.

- $$n_u > \frac{g_u}{g_\ell} n_\ell \implies T_{exc} < 0$$

  $$\frac{n_u}{n_\ell} \equiv \frac{g_u}{g_\ell} e^{-E_{u\ell}/kT_{ex}} \quad \text{OR} \quad T_{ex} \equiv \frac{E_{u\ell}/k}{\ln\left(\frac{g_u/g_\ell}{n_u/n_\ell}\right)}$$

- The stimulated emission is stronger than the absorption.

  $$\kappa_\nu = \sigma_{l\to u}\left(1 - \frac{n_u/n_\ell}{g_u/g_\ell}\right) < 0$$

  $\Rightarrow$ The radiation is <u>amplified</u> as it propagates!

These are bright and small in angular size, so masers have been used to measure proper motions in galaxies and near SMBHs.

Note: $\tau_\nu = \int_s^{obs} \kappa_\nu\, ds < 0 \Rightarrow$ so $e^{-\tau_\nu} > 1$.

Transfer Eqn. for $|kT_{ex}| \gg h\nu$

$$T_A \approx T_A(0)\,e^{|\tau_\nu|} + |T_{ex}|(1-e^{|\tau_\nu|})$$

$$\approx (T_A(0) + |T_{ex}|)\,e^{|\tau_\nu|} - T_{ex}$$

↑ Big + more peaked than $\tau_\nu \sim 1$.

# Resonance Lines II: Einstein Coefficients

- Two-level atom model [BB]
  - Transition probability (from the excited state) given by Einstein relation $A_{21} \sim 10^{8-9}$ $s^{-1}$ [BB]
  - $A_{21}$ calculated from the overlap of the wave functions for the initial and final states
  - Favors the bluer transition when there are multiple paths for decay, $A_{21} \sim$ (Dipole matrix element) * $\nu_{12}^3$
  - Einstein coefficients
  - Stimulated emission important when the upper level has a high population compared to the ground level
  - Probability of photon absorption (by an atom in the ground state) depends on the energy density of the electromagnetic field, $B_{12}U(\nu_{12})$, where $B_{12}$ is the Einstein coefficient for absorption

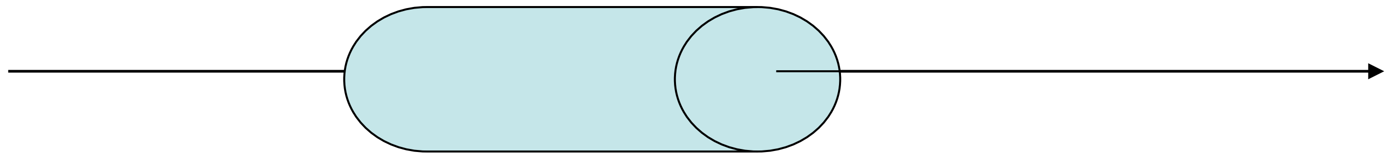
# Resonance Lines III: Oscillator Strength

- Conceptually useful to treat the active electron as oscillating between states.
- The "Oscillator Strength f" is the effective number of classical electrons involved in the transition.
- Strongest transitions have f~1.
- The sum of the f values for all the transitions in the atom cannot exceed the number of optically active electrons.

# Radiative Transfer

- Specific Intensity - constant along a ray
- Mean Intensity - zeroth moment
- Flux - 1st moment
- Radiation pressure - 2nd moment
- Quiz - Compute the flux from a uniformly bright source

# Resonance Line Transfer



- Basic equation [BB]
- Optical depth $\tau$ describes the reduction in intensity, $I=I_o e^{-\tau}$
- $\tau(\nu) = N\ \sigma(\nu)$, where N is column density.
- Linear absorption coefficient
- Absorption coefficient (per unit mass) is called **opacity** and includes contributions from scattering and absorption.

# Radiative Transfer

- Point source (star or quasar): Absorption is against a point source but emission is generally into the whole sphere. Fraction of emission towards the observer is negligible.

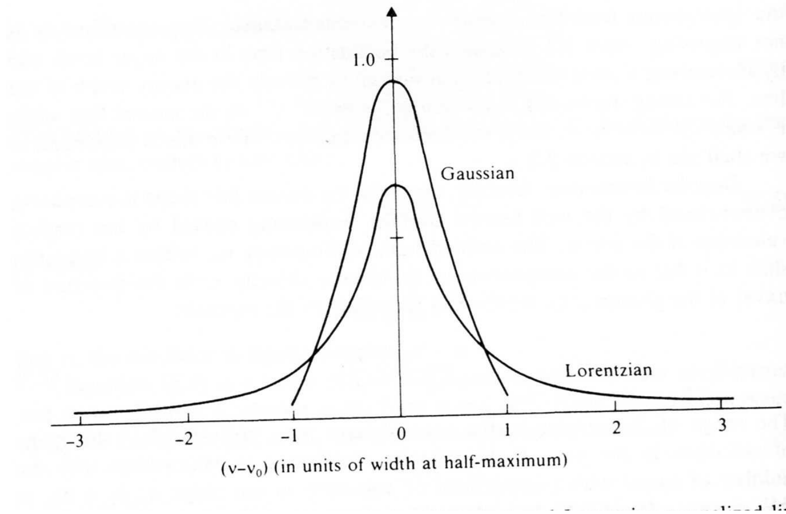
  [BB] $\mathbf{I_\nu(\tau) = I_{\nu,0}\exp(-\tau)}$

- Large beam (radio): Emission from all the atoms in the beam cannot be ignored.

  [BB - later]

  $\mathbf{I_\nu(\tau) = I_{\nu,s}\exp(-\tau) + \int j_\nu/\kappa_{\nu\,0}\exp(-[\tau - \tau'\ ])\, d\tau}$

# Line Profiles



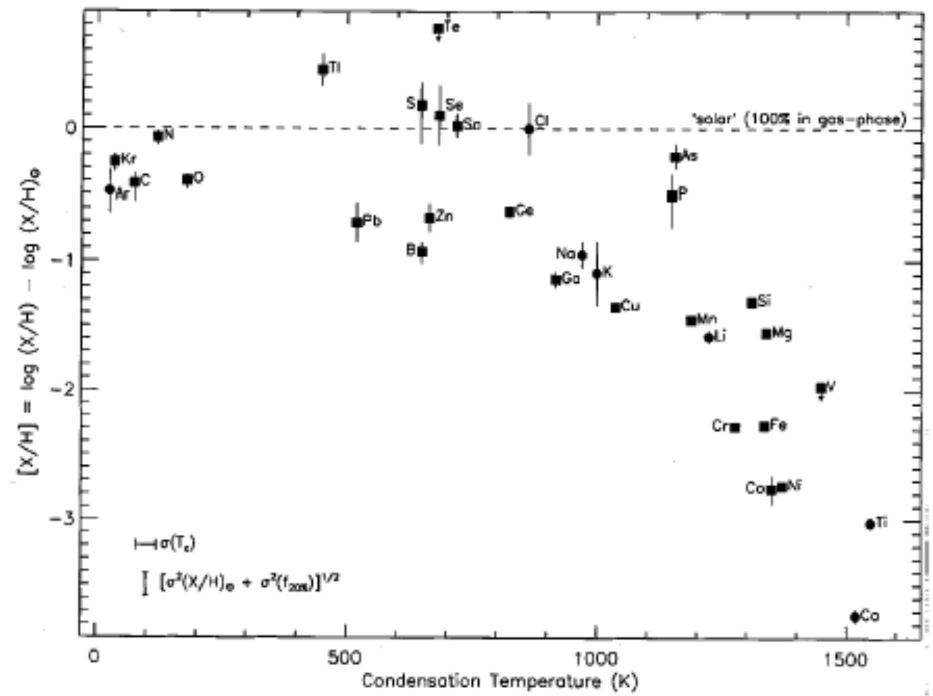

- Lorentzian - finite lifetime of excited state implies finite line width via Heisenberg uncertainty principle
  - Natural line width
  - Damping wings
- Pressure broadening - at higher densities collisional processes shorten lifetime of excited state ==> distribute lines photons over a larger energy range
- Doppler broadening - thermal motion
- Voigt Profile

# Applications of Column Density Measurements

- Useful for measuring chemical abundances
- Measured gas-phase depletion of elements
- Used for outflow/inflow measurements

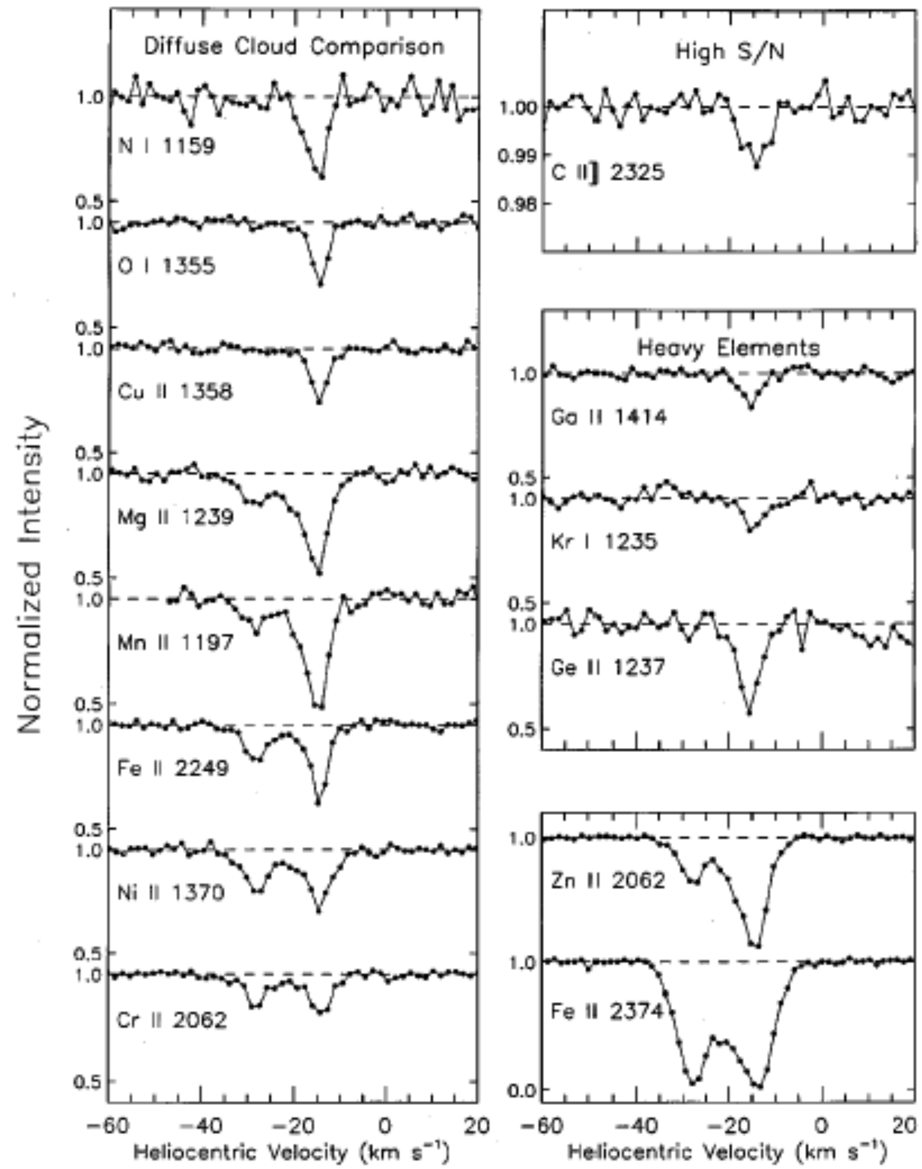
# Elemental Depletion

- Sightlines through the Milky Way disk and halo
- Savage & Sembach 1996 ARRA
- Hardly depleted (NI, OI), moderately depleted (MgII, MnII), and highly depleted (FeII, NiII, CrII)



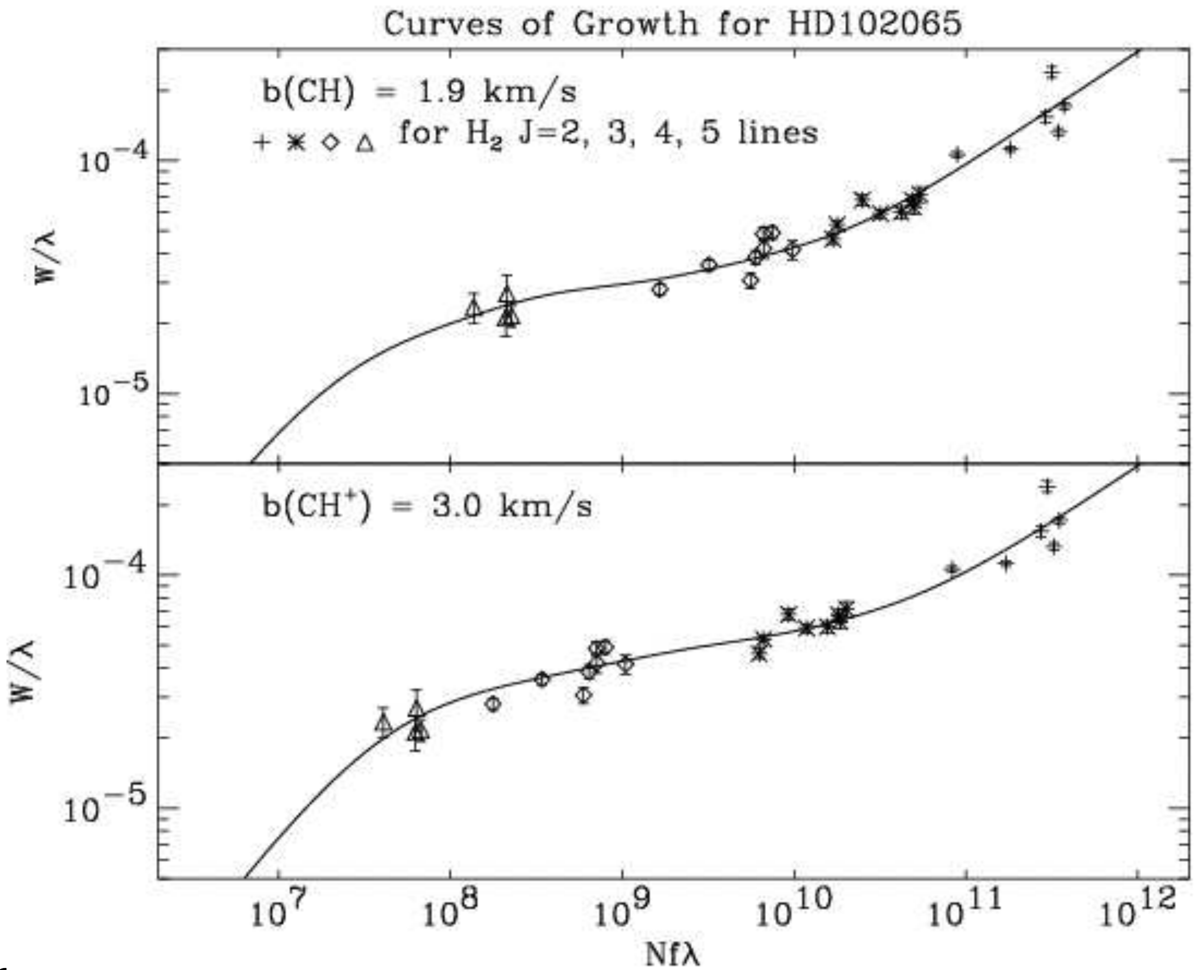
# Interstellar Abundances

- Cool diffuse cloud at v=-15 km/s with log (NHI) = 21.12
- Warm neutral cloud at v=-27 km/s with log (NHI) = 19.74
- HI column is 10 times smaller in the warm cloud
- Different conditions in the cool and warm clouds



# Curve of Growth

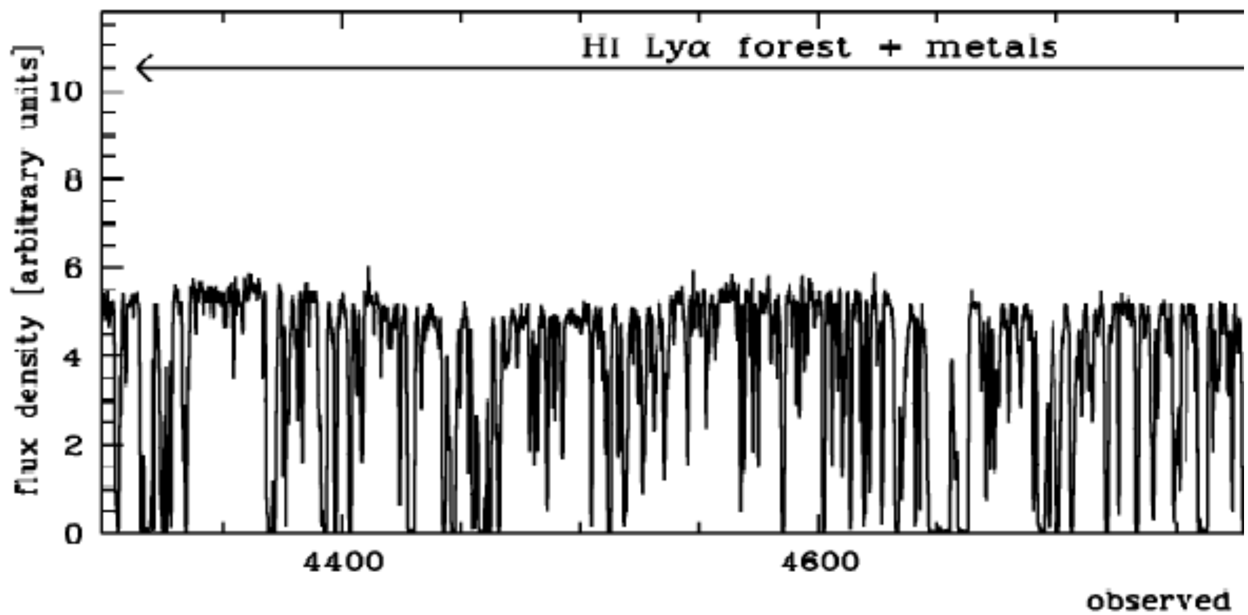
- Definition of **equivalent width W** [BB]



# Curve of Growth

- W propto N when $\tau(\nu) << 1$, the **linear portion** of the curve of growth
- W propto $(\ln N)^{1/2}$ when $\tau(\nu) \sim 1$,
  the **flat part** of the curve of growth
- W propot $(N)^{1/2}$ when $\tau(\nu) >> 1$,
  the **square root portion of the cog**

# Intervening Systems




- Damped Lya Systems, log N(HI) > 19
- Lyman Limit Systems, log N(HI) > 16.5
- Lya Forest, log N(HI) < 16

# Addendum (Equivalent Width)

- Intervening absorption line systems are generally not observed in the rest frame
- Remember your cosmology
- F(line) = L(line) / 4 pi D^2 (1+z)^2
- $F_\lambda = L_\lambda$ / 4 pi D^2 (1+z)^3
- Hence W(obs) = (1+z) * W(rest)

# Radiative Transfer with Wide Beam

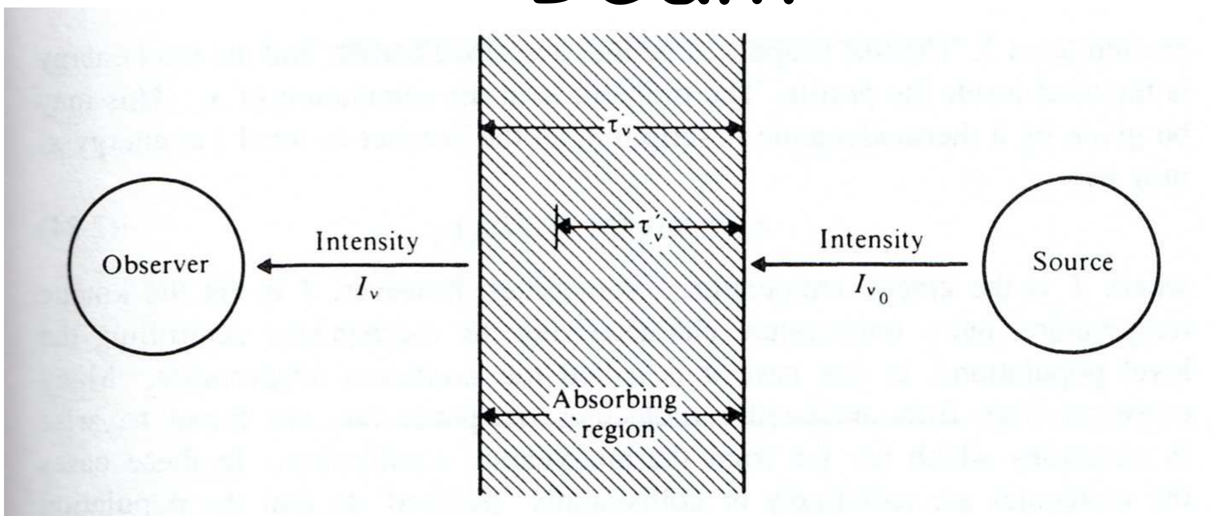



**Figure 2.3.** Schematic diagram defining intensities after absorption.

- Tranfer Eqn [BB]

$$I_\nu(\tau) = I_{\nu,S}\exp(-\tau) + \int j_\nu/\kappa_{\nu\,0}\exp(-[\tau - \tau'\ \ ])d\tau$$

# Brightness temperature $T_b$

- Define **brightness temperature** as the temperature a blackbody would have to have to give the observed intensity
- In the Rayleigh-Jeans limit (Radio Astronomy)

  $B(\nu,T) \sim 2kT\nu^2/c^2$ ; write radiative transfer in terms of T [BB]

# Implications of Transfer Eqn.

- Extremely optically thick line: the radiation field is limited by cloud's blackbody value, we don't see the source behind it
- Very thin cloud: See an attenuated source with little emission from cloud
- Absorption Line when $T(\nu) < T_{b,s}(\nu)$

# (Inside) Radio Astronomy

- Effective photosphere limits total line flux and prevents us from probing inside the cloud. An example $^{12}CO$ 115.271 GHz.
- Does the source fill the beam?

# HI 21-cm Line

- Ground level $1^2S_{1/2}$ splits into two hyperfine levels depending on whether the e- J is aligned or anti-aligned with the proton spin
- The (small) magnetic moment of the nucleus interacts with the magnetic field produced by the moving electron
- 1420.40575 MHz or 21.1049 cm
- So small the $T_s$ = T usually
- The orbital angular momentum QN l=0 for both states, so no E1
- A = 2.87e-15 s-1
- What is the natural linewidth?

# Astrophysical Masers

- Examples OH, H2CO
- Why is 3-2 transition rate higher than 2-1?
- $n2 = n1 \exp(-E_{ex}/kT)$
- Where T, excitation temperature, can be much (orders of magnitude) higher than the gas kinetic temperature.
- At molecular cloud temperatures of 10-100 K, the correction for stimulated emission can be significant for microwave lines.
- Intensity increases exponentially over a distance where the effective absorption coefficient is negative.

# Observations of Interstellar Masers

- What can we learn from masers?
- Amplification is largest close to line center, so the line profile becomes narrower than the Doppler width. *Good for measuring cloud motion and turbulence.*
- Amplification depends on rate of pumping, or the number of atoms in the masing beam. *Good for measuring local column density.*
- *Learn about the pump, which is usually a source such as a shock close to the masing volume.*