Emission and Absorption Coefficients

1) Emissivity

$$j_\nu = \frac{1}{4\pi} n_u A_{ul} h\nu \, \phi_\nu$$

line profile, describes the probability that the emitted photon will have frequency between ν and $\nu + d\nu$.

2) Attenuation Coefficient

describes the "net" absorption, i.e., true absorption minus stimulated emission.

$$K_\nu = n_l \sigma_{l \to u}(\nu) - n_u \sigma_{u \to l}(\nu)$$

$$= n_l \sigma_{l \to u} - \frac{n_u \sigma_{u \to l}}{n_l \sigma_{l \to u}} n_l \sigma_{l \to u}$$

$$= n_l \sigma_{l \to u}(\nu) \left[1 - \frac{n_u / n_l}{\sigma_{l \to u} / \sigma_{u \to l}} \right]$$

$$\frac{\sigma_{l \to u}}{\sigma_{u \to l}} = \frac{g_u}{g_l}$$

$$K_\nu = n_l \sigma_{l \to u}(\nu) \left[1 - \frac{n_u / n_l}{g_u / g_l} \right]$$

If the absorbers come into equilibrium with the radiation field, then the relative population of the levels depends on their energy difference, or equivalently, the photon energy.

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-h\nu/kT}, \text{ where } -h\nu = E_l - E_u$$

$$K_\nu = n_l \sigma_{l \to u}(\nu) \left[1 - e^{-h\nu/kT_{ex}} \right]$$

(a) stimulated emission can often be neglected because the upper levels of atoms and ions usually have negligible populations.

(b) at radio and sub-mm frequencies, it is important to include stimulated emission.

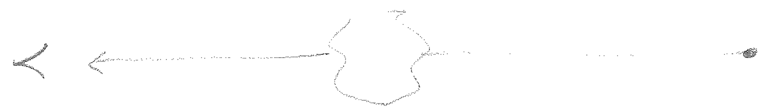
Radiative Transfer

$$dI_\nu = -\kappa I_\nu\, ds + j_\nu\, ds$$

"emissivity"

"attenuation coefficient"

(1) Point source (star or quasar): Absorption is against a point source but emission is generally into the whole sphere. Fraction of emission towards the observer is negligible.

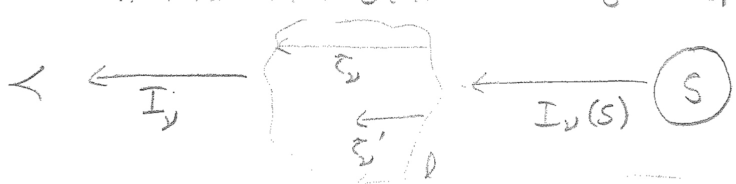


$$I_\nu^{-1}\, dI_\nu = -\kappa\, ds$$

$$I_\nu = I_\nu(0)\, e^{-\int \kappa\, ds}$$

$$I_\nu(\tau) = I_{\nu,0}\, e^{-\tau}$$

(2) Radio observations have large beam, emission from all the atoms in the beam cannot be ignored



Note: $\tau(x) \equiv \int_s^{obs} \kappa\, ds$

$$dI_\nu + I_\nu\, d\tau = \frac{j_\nu}{\kappa_\nu}\, d\tau \qquad \text{Transfer Eqn.}$$

<u>Integrating Factor</u> $e^{\int_{obs}^{l} \kappa\, ds}$

$$e^{\int_{obs}^{l} \kappa ds}\, dI_\nu + e^{\int_{obs}^{l} \kappa ds}\, I_\nu\, d\tau = j/\kappa\; e^{\int_{obs}^{l} \kappa ds}\, d\tau$$

Note Limits

$$\int_s^{obs} d\left[e^{\int_{obs}^{l} \kappa ds}\, I_\nu\right] = \int_0^{\tau_\nu} j/\kappa\; e^{\int_{obs}^{l} \kappa ds}\, d\tau$$

$$e^{\int_{obs}^{obs} \kappa ds}\, I_\nu - e^{\int_{obs}^{s} \kappa ds}\, I_\nu(s) = \quad ''$$

$$I_\nu = e^{-\tau_\nu} I_\nu(s) + \int_0^{\tau_\nu} j/\kappa\; \exp\left[\int_{obs}^{s} \kappa ds - \int_l^{s} \kappa ds\right] d\tau'$$

$$\boxed{I_\nu = e^{-\tau_\nu} I_\nu(s) + \int_0^{\tau_\nu} j_\nu/\kappa_\nu\; e^{-[\tau_\nu - \tau_\nu']}\, d\tau'}$$

The source and its attenuation

... Each source in the cloud and the attenuation between that point in the cloud and the observer.

If cloud in LTE, then Kirchhoff's Law applies

$$j(\nu) = K(\nu)\, B_\nu(T),$$

i.e., emission and absorption are equal for a blackbody.

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

$$I_\nu = I_\nu(s)\, e^{-\tau_\nu} + B_\nu(T) \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')}\, d\tau'$$

Let $x = -(\tau_\nu - \tau_\nu') = \tau_\nu' - \tau_\nu$

$dx = d\tau_\nu'$

$$\int_{-\tau_\nu}^{0} e^x\, dx = e^0 - e^{-\tau_\nu}$$

$$I_\nu = I_\nu(s)\, e^{-\tau_\nu} + B_\nu(T)\left(1 - e^{-\tau_\nu}\right)$$

e.g., When cloud is very diffuse $\tau_\nu \approx 0$, and we get no emission from the cloud

$$B_\nu(T)\left(1 - e^{-\tau_\nu}\right) \longrightarrow B_\nu(T)\,(1-1) \approx 0$$

When cloud is very thick, $\tau_\nu \gg 1$, $B_\nu(T)(1-0) \rightarrow B_\nu(T)$
then cloud radiates as a BB.

And we do not see the source behind it

$$\lim_{\tau_\nu \to \infty} I_\nu(s)\, e^{-\tau_\nu} \Rightarrow 0$$

Radio Astronomy

$$\frac{h\nu}{kT} << 1 \implies e^{h\nu/kT} - 1 \approx 1 + \frac{h\nu}{kT} - 1 + \cdots \approx \frac{h\nu}{kT}$$

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2}\,\frac{kT}{h\nu} \approx \frac{2\nu^2 kT}{c^2}$$ Rayleigh-Jeans Law

$B_\lambda = B_\nu \cdot \frac{c}{\lambda^2}$

$$I_\nu = I_\nu(s)\, e^{-\tau_\nu} + \frac{2\nu^2 kT}{c^2}\left(1 - e^{-\tau_\nu}\right)$$

Define antenna temperature $T_A \ni$

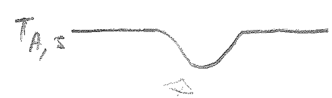
$$\frac{c^2 I_\nu}{2\nu^2 k} = I_\nu(s)\,\frac{c^2}{2\nu^2 k}\, e^{-\tau_\nu} + T\left(1 - e^{-\tau_\nu}\right)$$

$$\boxed{T_A = T_{A,s}\, e^{-\tau_\nu} + \underbrace{T}_{\text{cloud Temperature}}\left(1 - e^{-\tau_\nu}\right)}$$

(a) $T_{A,s} > T \implies$ Absorption Line

(i) $\tau_\nu << 1$ at line center

$$T_A = T_{A,s}(1 - \tau_\nu) + T\left(1 - (1 - \tau_\nu)\right)$$
$$= T_{A,s} - (T_{A,s} - T)\,\tau_\nu$$



(ii) $\tau_\nu > 1$ at line center, the $e^{-\tau_\nu} \to 0$

and $T_A \approx T$ see cloud not the source at line center

but $T_A > T$ away from line center

Radio Astronomy

- Brightness temperature is defined to be the temperature such that a blackbody at that temperature would have specific intensity $B_\nu(T_{BB}) = I_\nu$.
- In LTE, the brightness temperature is the actual thermodynamic temperature.

$$B_\nu(T) \equiv \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

so $$T_B(\nu) \doteq \frac{h\nu}{k} \ln\left[\frac{2h\nu^3}{c^2 I_\nu} + 1\right]^{-1}$$

- T_B is a non-linear function of temperature.

- Astronomers define antenna temperature so that it is linear in the intensity. At radio frequencies, $h\nu \ll kT$, and $B_\nu(T) \approx \frac{2h\nu^3}{c^2}\frac{kT}{h\nu}$.

Hence, $$T_A \equiv \frac{c^2}{2k\nu^2} I_\nu$$

Radio Astronomy - Maser Lines

Upper levels are populated at radio and sub-mm frequencies.

(1) Include spontaneous emission

$$I_\nu = I_\nu(0)\, e^{-\tau_\nu} + \underline{B_\nu(T_{ex})(1-e^{-\tau_\nu})}$$

(2) Include stimulated emission

$$K_\nu = n_l\, \sigma_{l\to u} (1 - e^{-h\nu/kT_{ex}})$$

Collisional or radiative excitation of a level higher than u that then decays to populate level u causes a population inversion when the pumping is rapid compared to the processes that depopulate level u.

- $n_u > \frac{g_u}{g_l} n_l \implies T_{exc} < 0$

$$\frac{n_u}{n_l} \equiv \frac{g_u}{g_l} e^{-E_{ul}/kT_{ex}} \quad \text{OR} \quad T_{ex} \equiv \frac{E_{ul}/k}{\ln\left(\frac{g_u/g_l}{n_u/n_l}\right)}$$

- The stimulated emission is stronger than the absorption.

$$K_\nu = \sigma_{l\to u}\left(1 - \frac{n_u/n_l}{g_u/g_l}\right) < 0$$

$\Rightarrow$ The radiation is amplified as it propagates!

These are bright and small in angular size, so masers have been used to measure proper motions in galaxies and near SMBHs.

Note: $\tau_\nu = \int_s^{obs} K_\nu\, ds < 0 \implies$ so $e^{-\tau_\nu} > 1$.

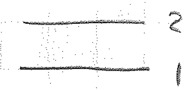
Transfer Eqn. for $|kT_{ex}| \gg h\nu$

$$T_A \approx T_A(0)\, e^{|\tau_\nu|} - |T_{ex}|(1 - e^{|\tau_\nu|})$$

$$\approx (T_A(0) + |T_{ex}|)\, e^{|\tau_\nu|} - |T_{ex}|$$

$\uparrow$ Big + more peaked than τ_ν ~~1~~ 1.

Emissivity & Absorption Coefficients



$$dI_\nu = j_\nu \, ds - k_\nu I_\nu \, ds$$

Expressed in terms of atomic constants

1) $$\int_{\substack{\text{Line}\\\text{Profile}}} j_\nu \, d\nu = \frac{n_2 A_{21} h\nu_{21}}{4\pi}$$

2) Total energy absorbed in a line must be $\int k_\nu I_\nu \, ds$, but it must also be the difference between stimulated absorption and stimulated emission.

$$\int_{\substack{\text{Line}\\\text{Profile}}} k_\nu I_\nu \, d\nu = [n_1 B_{12} - n_2 B_{21}] \frac{U(\nu_{12})}{4\pi} h\nu_{21}$$

Recall energy density $= \frac{4\pi}{c} \times$ Mean Intensity

$$I_\nu \int k_\nu \, d\nu = [n_1 B_{12} - n_2 B_{21}] \frac{h\nu_{21}}{4\pi} \frac{4\pi}{c} I_\nu$$

And Doppler width >> Natural Line width

$$\therefore \quad \int k_\nu \, d\nu = [n_1 B_{12} - n_2 B_{21}] \frac{h\nu_{21}}{c}$$

Note:

Emissivity depends on population of upper level whereas

Absorption depends on population difference between two levels

Now consider the amount of absorption per ion/particle.

That will allow us to connect τ to the amount of material along the L.O.S.

Define $s_\nu = \frac{K_\nu}{n_1} = s\,\phi(\nu)$

Area = Absorption coefficient per particle

Integrate along l.o.s.

$$\int n_1 s\,\phi(\nu)\,ds = \int K_\nu\,ds$$

$$\int n_1 s\,\phi(\nu)\,ds = \tau_\nu$$

$$\langle\phi(\nu)\rangle_{LOS} \int n_1 s\,ds = \tau_\nu$$

Average line profile along sightline

$$\langle\phi(\nu)\rangle\, s \int n_1\,ds = \tau_\nu$$

$$\langle\phi(\nu)\rangle\, s\, N_1 = \tau_\nu$$

Define column density

When we integrate τ_ν over the line profile

$$\tau = \int \tau_\nu\,d\nu = \int \langle\phi(\nu)\rangle s N_1\,d\nu$$

$$= s N_1 \int \langle\phi(\nu)\rangle\,d\nu$$

$$* \qquad \tau = s N_1$$

Absorption Cross-section per particle × # particles per unit area

And we can express "s" in terms of atomic constants.

$$\int n_1 s\,\phi(\nu)\,d\nu = \int K_\nu\,d\nu \qquad \text{by def. of } s$$

$$n_1 s \underbrace{\int \phi(\nu)\,d\nu}_{=1} = \left[n_1 B_{12} - n_2 B_{21}\right] \frac{h\nu_{21}}{c} \qquad \text{From page 1}$$

$$s = \left[B_{12} - \frac{n_2}{n_1} B_{21}\right] \frac{h\nu_{21}}{c}$$

$$= \frac{h\nu_{21}}{c} B_{12} \left[1 - \frac{n_2}{n_1} \frac{B_{21}}{B_{12}}\right]$$

In LTE

(i) $\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu_{21}/kT}$ Boltzmann

(ii) $g_1 B_{12} = g_2 B_{21}$ Detailed Balance

$$\Rightarrow \quad s = \frac{h\nu_{21}}{c} B_{12} \left[1 - e^{-h\nu_{21}/kT} \right]$$ $\frac{g\ cm^3}{s^2} \frac{s}{cm}$

↑ correction for stimulated emission

For resonance lines, $h\nu \gg kT$ almost always in ISM

$$s \Rightarrow s_{uncorrected} = \frac{h\nu_{21}}{c} B_{12}$$

Relate "s" to oscillator strength "f".

Using $B_{12} = \frac{c^3}{8\pi h} \frac{1}{\nu_{12}^3} A_{12}$ Einstein Relation

And $A_{12} = \frac{8\pi^2 e^2}{m_e c^3} \nu_{12}^2 f_{12}$ Previous Def. f

Reduces to

$\frac{g\ cm^2 \cdot cm}{s^2} \frac{s^3}{g\ cm^3} \frac{1}{s^2} = \frac{1}{s}$

So f is dimensionless

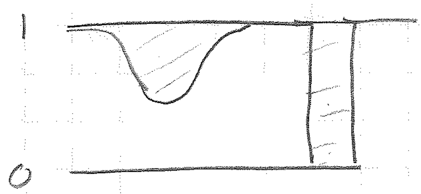
$$s_u = \frac{h\nu_{21}}{c} \frac{c^3}{8\pi h} \frac{1}{\nu^3} \frac{8\pi^2 e^2}{m_e c^3} \nu^2 f_{12}$$

$$s_u = \frac{\pi e^2}{m_e c} f_{12}$$

$$f_{12} = \frac{m_e c}{\pi e^2} s_u = \frac{s_u}{2.654 \times 10^{-2}\ (cm^2)}$$ [Hz]

$$g \frac{cm}{s} \frac{s^2}{g\ cm^2 \cdot cm} cm^2 = s = Hz^{-1}$$

Equivalent Width + Curve of Growth



$$W = \int_{-\infty}^{\infty} \frac{I_{\nu,c} - I_\nu}{I_{\nu,c}}\, d\nu$$

$\lambda d\nu + \nu d\lambda = 0$

$d\nu = -\frac{c\, d\lambda}{\lambda^2}$

Note: $W_\lambda = W_\nu \frac{\lambda^2}{c}$

And $W = \int_{-\infty}^{\infty} 1 - e^{-\tau_\nu}\, d\nu$

i) Weak Lines $\tau_\nu << 1$

$$W_\nu = \int 1 - e^{-\tau_\nu}\, d\nu$$

$$\approx \int 1 - (1 - \tau_\nu + \cdots)\, d\nu$$

$$\approx \int \tau_\nu\, d\nu$$

$$= \left[\frac{\pi e^2}{m_e c} f_{12}\right] N_1$$

$$W_\lambda = \frac{\pi e^2}{m_e c^2} \lambda^2 f_{12} N_1$$

$$N_1 = 1.13 \times 10^{20}\ \mathrm{cm}^{-2}\ \frac{W_\lambda}{f_{12} \lambda^2}$$

W_λ and λ in Å

W Linear in Column Density!

What about the more general case when $\tau \sim 1$?

ii) General Case – Curve of Growth

Functional dependence of W_λ / λ on $N_1 \lambda f_{12}$ depends on the line profile $\phi(\Delta\nu)$.

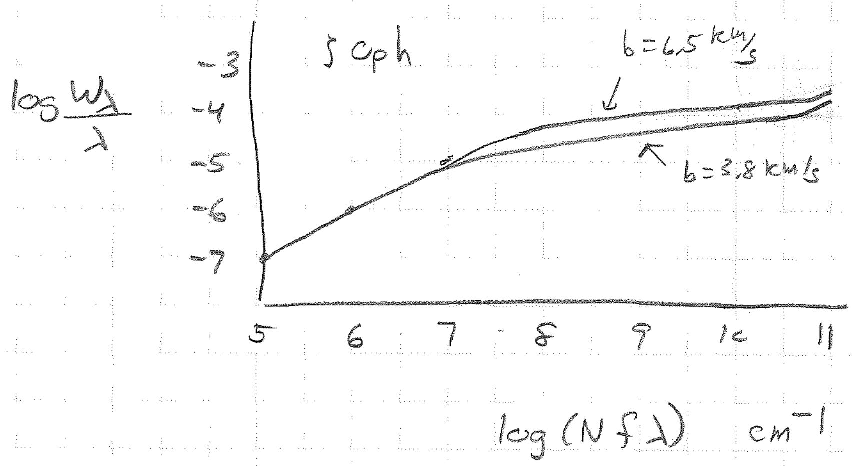
$$W_\lambda = \frac{\lambda^2}{c} \int 1 - e^{-\tau_\nu} \, d\nu$$

where $\tau_\nu = s N_1 \langle \phi(\nu) \rangle_{los}$

Can integrate this analytically for any line shape.

For the usual Voigt profile, we get 3 regimes –

Linear part of cog
Flat part part of cog
Square-root part of cog



Spitzer 3.2

Natural Line Profile

Normalized Line Profile $\phi(\nu)$: $s = \sigma_{12}(\nu) = \frac{\pi e^2}{m_e c} f_{12}\, \phi_\nu$, $\int \phi_\nu\, d\nu \equiv 1$

σ_{12}: absorption cross section

Lorentz line profile : $\boxed{\phi_\nu = \frac{4\gamma_{21}}{16\pi^2(\nu-\nu_{21})^2 + \gamma_{21}^2}}$, $\nu_{12} = \frac{E_2 - E_1}{h}$

$\Delta\nu\,(\text{FWHM}) = \frac{\gamma_{21}}{2\pi}$ ← Narrow or Wide?

$\Delta V = \frac{\Delta\nu}{\nu_{21}} c = 0.0121 \frac{\text{km}}{\text{s}} \left(\frac{\lambda_{21}\,\gamma_{21}}{7618\ \text{cm s}^{-1}}\right)$, For Ly α

$$\boxed{\begin{aligned}&\text{Morton: } \gamma_u = \tfrac{AB}{\text{GAMMA}} = 6.265\times10^{8}\,\text{s}^{-1}\\ &\frac{\gamma_{21}}{2\pi}\lambda_{21} = \frac{6.265\times10^{8}\,\text{s}^{-1}(1215.6737)10^{-8}}{2\pi \quad 10^{5}\,\text{cm/km}}\\ &= 0.0121\,\tfrac{\text{km}}{\text{s}}\left(\frac{\gamma_{21}\lambda_{21}}{7616\,\text{cm s}^{-1}}\right)\end{aligned}}$$

Heisenberg uncertainty principle $\Delta E \Delta t \geq \hbar$: An energy level has a width ΔE inversely proportional to the lifetime of the level.

True Line Profile

$\phi_\nu = \phi_\nu^{\text{Lorentz}} \circ \phi_\nu$ (velocity distribution of absorbers)

When the velocities are entirely due to thermal motions (i.e., particle energies are M.B.), then the 1D velocity distribution is Gaussian. The absorption line will have a Voigt profile:

$$\phi_\nu^{\text{Voigt}} \equiv \frac{1}{\sqrt{2\pi}} \int \frac{dv}{\sigma_v} e^{-v^2/2\sigma_v^2} \underbrace{\frac{4\gamma_{21}}{16\pi^2[\nu-(1-v/c)\nu_{21}]^2 + \gamma_{21}^2}}_{\text{The natural profile at each velocity}} .$$

where Doppler parameter $b = \sqrt{2}\,\sigma_v$

Two Regimes:

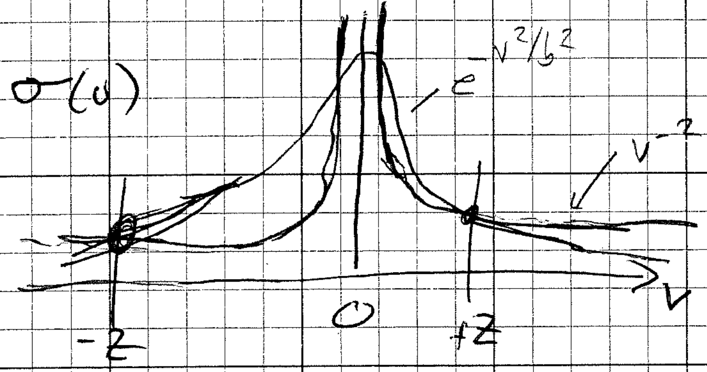
(1) "Near" line center ⇒ Doppler core profile $[\sigma_v >> |\Delta v|_{\text{natural}}]$

$$s \approx \sqrt{\pi} \frac{e^2}{m_e c} \frac{f_{12}\lambda_{21}}{b} e^{-v^2/b^2}$$

(2) "Large" frequency shifts ⇒ damping wings

$$s \approx \sqrt{\pi} \frac{e^2}{m_e c} \frac{f_{12}\lambda_{21}}{b} \left[\frac{1}{4\pi^{3/2}} \times \frac{\gamma_{21}\lambda_{21}}{b} \frac{b^2}{v^2}\right]$$

How far from line center is the transition from the Doppler core to the damping wings?



At $z \equiv v/b$, they cross.

$$e^{-z^2} = \frac{1}{4\pi^{3/2}} \frac{\gamma_{21}\lambda_{21}}{b} \frac{b^2}{v^2}$$

$$\boxed{e^{z^2} = 4\pi^{3/2} \frac{b\, z^2}{\gamma_{21}\lambda_{21}}}$$

$$e^{z^2} = 2924 \left[\frac{7618\ \mathrm{cm\,s^{-1}}}{\gamma_{21}\lambda_{21}} \left(\frac{b}{10^6\ \mathrm{cm/s}} \right) \right] z^2$$

The solution to this transcendental equation is

$$z^2 \simeq 10.31 + \ln\left[\frac{7618\ \mathrm{cm\,s^{-1}}}{\gamma_{21}\lambda_{21}}\, b_6 \right]$$

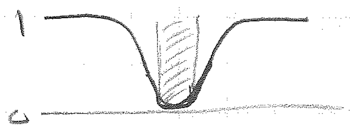
provided the quantity in brackets is not very large or very small.

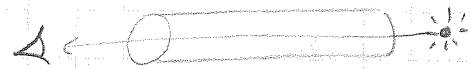
For H Lyα, the damping wings dominate for

$$\left|\frac{v}{b}\right| \geq 3.2, \text{ or}$$

$$|\Delta v| \geq 3.2\times10^6\, b_6 = 32\ \mathrm{\frac{km}{s}}\ \frac{b}{10^6\ \mathrm{cm/s}}$$

a) $\tau_\nu \gtrsim 1$ in the core of the line profile





All photons absorbed in the core, so the line cannot get much stronger even if # of atoms increased.

$\Rightarrow$ "flat" portion of cog

EW can increase a little due to absorption in Doppler wings of the profile.

$$\phi(\Delta\nu) = \frac{\lambda_{12}}{\sqrt{\pi}\, b} e^{-(v/b)^2}$$

Ignore Damping Wings

<u>Doppler Parameter</u> $b \equiv \sqrt{\frac{2kT}{m}}$

$$v = \frac{\nu - \nu_{12}}{\nu_{12}} c = \frac{\Delta\nu}{\nu_{12}} c$$

Note: $b = \sqrt{2}\,\sigma$

HWHM: $\ln(0.5) = -\left(\frac{HWHM}{b}\right)^2$

$HWHM = b\sqrt{\ln 2}$

$$\boxed{FWHM = 2\sqrt{\ln 2}\; b}$$

Let $x = v/b$ and $dx = \frac{1}{b} dv = \frac{c}{b}\frac{d\nu}{\nu_{12}}$

Define $x_1 \ni \tau(x_1) = 1$

$\tau_\nu = \tau_0 e^{-x^2}$, where τ_{ν_0} is optical depth at line center.

$1 = \tau_0 e^{-x_1^2}$

$x_1 = \sqrt{\ln \tau_0}$

$$1 - e^{-\tau_\nu} \simeq \begin{cases} 1 & \text{for } |x| < x_1 \\ \tau_\nu & \text{for } |x| >> x_1 \end{cases}$$

$$\frac{W_\lambda}{\lambda_{12}} = \frac{\lambda_{12}}{c}\,\nu_{12}\,\frac{b}{c}\int_0^\infty 1 - e^{-\tau(x)}\,dx \quad \times\ 2$$

$$\approx \frac{2b}{c}\left[\int_0^{x_1} dx + \int_{x_1}^\infty \tau_0\, e^{-x^2}\,dx\right]$$

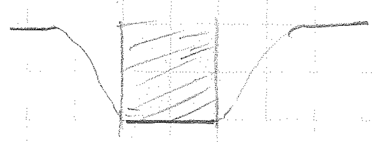
$$\simeq \frac{2b}{c}\left[\sqrt{\ln \tau_0} + \tau_{\nu_0}\left\{\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x_1)\right\}\right]$$

$$\simeq \frac{2b}{c}\left[\sqrt{\ln \tau_0} + \underbrace{\tau_{\nu_0}\frac{\sqrt{\pi}}{2}\left\{1 - \mathrm{erf}\left(\sqrt{\ln \tau_0}\right)\right\}}_{\lim_{\tau_{\nu_0}\to\infty} \text{ term} \Rightarrow \emptyset}\right]$$

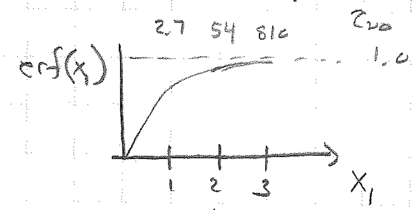
using

$$\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^{x_1} e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(x_1)$$



$-x_1$ $+x_1$



Area of core depends linearly on Doppler parameter

$\tau_{\nu_0} = e^{x_1^2}$

$$\boxed{\frac{W_\lambda}{\lambda} = \frac{2b}{c}\sqrt{\ln \tau_0}} \qquad \propto \sqrt{\ln N}$$

$$\tau_0 = s N_1\, \phi(\Delta\nu = 0)$$

$$= \frac{s\, N_1\, \lambda_{12}}{\sqrt{\pi}\, b}$$

$$= \frac{\pi e^2}{m_e c} f_{12} \frac{N_1}{\sqrt{\pi}\, b}\lambda_{12}$$

$$= \frac{1.497 \times 10^{-10}\, N_1\, \lambda_{12} f_{12}}{b}$$

cm^{-2} Å

km/s

$$\tau_0 = 1 \Rightarrow N = 8 \times 10^{13}\ \frac{b}{10\,\mathrm{km/s}}\ \frac{1}{f}\ \frac{1216\,\text{Å}}{\lambda}$$

Strong line $f \simeq 1$
$b = 10\,\mathrm{km/s}$

b) $\tau_o > 10^4$ Damping Wings Important

Natural Line width $\quad \phi(\nu) \propto \dfrac{1}{\Delta\nu^2 + (\Gamma/4\pi)^2}$

is very narrow because radiation damping constant Γ related to transition probability; and $A \sim 10^8\ s^{-1}$.

A^{-1} measures the occupation time in the upper level, and Heisenberg's uncertainty relation gives energy width

$$\Delta E \gtrsim \frac{h}{t} \approx \frac{6.63 \times 10^{-27}\ \text{erg} \cdot s}{10^{-8}\ s} \approx 6.6 \times 10^{-19}\ \text{erg}$$

$\Gamma = 4\pi \left(\frac{1}{4\pi} \sum_m A_{km} \right)$

$$E \approx \frac{hc}{\lambda} \approx \frac{12400\ eV \cdot \mathring{A}}{1216\ \mathring{A}} \left(1.6 \times 10^{-12} \tfrac{erg}{eV}\right) \approx 10\ eV \approx 1.6 \times 10^{-11}\ \text{erg}$$

$\text{HWHM} = \dfrac{\Gamma}{4\pi}$

$$\Delta E / E \sim \frac{10^{-18}}{10^{-11}} \sim 10^{-7}$$

$\dfrac{A}{0.5A} = \dfrac{(\Gamma/4\pi)^{-2}}{\frac{1}{\Delta w^2 + (\Gamma/4\pi)^2}} = \dfrac{\text{HWHM}^2 + (\Gamma/4\pi)^2}{(\Gamma/4\pi)^2}$

$$\Delta v \sim 10^{-7} c \approx 0.03\ km/s$$

Voigt Profile = Lorentzian $\circ$ Doppler

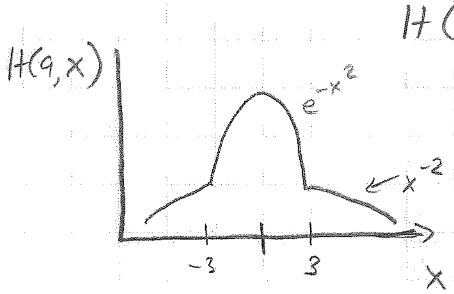
$$\phi(x) = \frac{\lambda_{12}}{\sqrt{\pi}\, b} H(a, x)$$

$$x = \frac{w}{b} \text{ again and } a \equiv \frac{\Gamma}{4\pi} \frac{c}{b \nu_o}$$

$$H(a, x) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(x-y)^2 + a^2}\, dy$$

$$H(a, x) \approx e^{-x^2} + \frac{a}{\sqrt{\pi}} \frac{1}{x^2 (+a^2)}$$

@ small x $\qquad$ @ large x



$$\phi(0) = \frac{\lambda_{12}}{b}$$

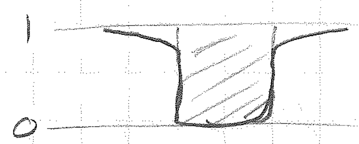
Again $x_1 \ni \tau(x_1) = 1$ $\qquad dx = \frac{c}{b}\frac{d\nu}{\nu_{12}}$

$$1 - e^{-\tau_\nu} = \begin{cases} 1 & |x| \le x_1 \\ \tau(x) = \tau_0 \frac{a}{\sqrt{\pi}} x^{-2} & |x| > x_1 \end{cases}$$

$$\frac{\tau_0 a}{\sqrt{\pi}} = x_1^2$$

$$\frac{W_\lambda}{\lambda_{12}} = \frac{\lambda_{12}}{c}\,\frac{b\,\nu_{12}}{\cancel{c}} \int_{-\infty}^{\infty} 1 - e^{-\tau_\nu}\, dx$$

$$= 2\frac{b}{c}\left[\int_0^{x_1} dx + \int_{x_1}^{\infty} \frac{\tau_0 a}{\sqrt{\pi}} x^{-2} dx\right]$$



$$\int x^{-2} dx = -x^{-1}$$

$$\int_{x_1}^{\infty} x^{-2} dx = \frac{-1}{\infty} + \frac{1}{x_1}$$

$$\frac{W_\lambda}{\lambda_{12}} = \frac{2b}{c}\sqrt{\frac{a\tau_0}{\sqrt{\pi}}} + \frac{2b}{c}\,\frac{\tau_0 a}{\sqrt{\pi}}\sqrt{\frac{\sqrt{\pi}}{a\tau_0}}$$

$$\frac{W_\lambda}{\lambda} = \frac{2b}{c}\sqrt{\frac{a\tau_0}{\sqrt{\pi}}} \times 2$$

where $\tau_0 = s N_1 \phi(\Delta\nu = 0) = \frac{s N_1 \lambda_{12}}{\sqrt{\pi}\, b}$

$$= \left[\frac{\pi e^2}{m_e c} f_{12}\right] N_1 \frac{\lambda_{12}}{\sqrt{\pi}\, b}$$

$$\frac{W_\lambda}{\lambda} = \frac{2\cancel{b}}{c}\sqrt{\left[\frac{\Gamma}{4\pi}\,\frac{\lambda_{12}}{\cancel{b}}\right] s N_1 \frac{\lambda_{12}}{\cancel{b}}} \quad \times \frac{2}{\sqrt{\pi}}$$

This factor is wrong.

$$\boxed{\frac{W_\lambda}{\lambda} = \frac{2}{c}\sqrt{\frac{\Gamma}{4\pi}\lambda_{12}^2\, s N_1}}$$

← Spitzer 3-53

Agrees with Wolfe et al 1986 ApJS 61, 249 Eqn. 3

← $W_\lambda = 7.3\,\text{Å}\sqrt{N_{20}}$

week 4

HI 21-cm Emission & Absorption

Another way to detect and study neutral hydrogen, the primary component of the ISM, is the 21-cm line.

- Hyperfine splitting of 1s ground level produced by coupling of e^-'s magnetic moment to the magnetic field produced by the magnetic moment of the proton's spin.

2 ———— $\hat{\Phi}^{p}\hat{\Phi}^{e^-}$ $s=1$ $E_2 = 5.87\times10^{-6}$ eV

$A_{12} = 2.8843\times10^{-15}\ s^{-1}$

$\nu = 1420.4$ MHz

1 ———— $\hat{\Phi}\ \underset{\vee}{\Phi}$ (p, e^-) $s=0$ $E_1 = 0$

- When the e^- drops to the ground state, its spin flips, and a photon is emitted.

N.B. $\dfrac{\Delta E}{k} = 0.06816\ K \Rightarrow$ CMB populates upper state

We expect $\dfrac{h\nu_{12}}{kT_{ex}} \ll 1$ everywhere.

Population of Upper Level:

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu_{12}/kT_{ex}}$$

$$= \frac{2(1)+1}{2(0)+1} e^{-0.0682K/T_s}$$

$$\simeq 3$$

(1) We call the "excitation temperature" the "spin temperature."

(2) 75% of the HI atoms have the e^- in the upper level.

$$n_2 \simeq \frac{3}{4} n(HI) \quad \text{and} \quad n_1 \simeq \frac{1}{4} n(HI)$$

21-cm Emissivity:

$$j_\nu = n_2 \frac{A_{21}}{4\pi} h\nu_{21}\ \phi(\nu)$$

$$= \frac{3}{16} A_{21} h\nu_{21}\ \phi(\nu)$$

21-cm Absorption

21-cm Attenuation coefficient:
$$\kappa_\nu = n_1 \frac{g_2}{g_1} \frac{A_{21}}{8\pi} \lambda_{21}^2 \left[1 - e^{-h\nu_{21}/kT_s}\right] \phi(\nu)$$

$\Rightarrow$ Correction for stimulated emission is important!

$$e^{-h\nu_{21}/kT} \approx 1.$$

$$\kappa_\nu \approx n_1 \frac{g_2}{g_1} \frac{A_{21}}{8\pi} \lambda_{21}^2 \left[\frac{h\nu_{21}}{kT_s}\right] \phi(\nu)$$
$$\approx \frac{1}{4} n(HI) \frac{3}{8\pi} A_{21} \lambda_{21}^2 \frac{h\nu_{21}}{kT_s} \phi(\nu)$$
$$\approx \frac{3}{32\pi} A_{21} \frac{hc\lambda_{12}}{kT_s} n(HI) \phi(\nu)$$

$\Rightarrow$ N.B. Absorption coefficient $\kappa_\nu \propto T_s^{-1}$.

Absorption Optical Depth:
$$\tau_\nu = \int_s^{obs.} \kappa_\nu \, ds$$

Consider H atoms with a Gaussian velocity distribution

$$\tau_\nu = \frac{3}{32\pi} A_{21} \frac{hc}{kT_s} \lambda_{12} \frac{1}{\sqrt{\pi}} \frac{c}{\nu_{21}} \frac{1}{b} e^{-v^2/b^2} \int_s^{obs} n(HI)\, ds$$

$b = \sqrt{2}\sigma$
$$= 2.190 \frac{N(HI)}{10^{21}\,cm^{-2}} \left[\frac{100K}{T_s}\right] \frac{km/s}{(b/\sqrt{2})} e^{-v^2/b^2}$$

Some regions in the ISM have $N(HI) \gtrsim 10^{21}\,cm^{-2}$ and $T_s \approx 100K$, so self-absorption in the 21-cm line can be important!

21-cm Emission & Gas Mass

Optically Thin Cloud ($\tau_\nu \lesssim 0.2$): $N(HI) \lesssim 10^{20}\ cm^{-2} \left[\frac{T_s}{100K}\right]\left[\frac{b/\sqrt{2}}{km/s}\right]$

Transfer Eqn.: $I_\nu = I_\nu(0) + \int j_\nu ds$

$$I_\nu - I_\nu(0) = \frac{3}{16\pi} A_{21} h\nu_{21} \phi_\nu N(HI)$$

Integration over Line Profile: $\int I_\nu - I_\nu(0)\, d\nu = \frac{3}{16\pi} A_{21} h\nu_{21} N(HI)$

Expressed in terms of antenna temperature $T_A = \frac{c^2 I_\nu}{2k\nu^2}$ and relative velocity $\nu = \nu_{21}(1 - v/c)$

$$\int \frac{c^2}{2k\nu^2}\left[I_\nu - I_\nu(0)\right]\frac{c}{\nu_{21}} d\nu = \frac{c^2}{2k\nu^2} \frac{3}{16\pi} A_{21} h\cancel{\nu_{21}} N(HI) \frac{c}{\cancel{\nu_{21}}}$$

$$\int T_A - T_A(0)\, dv = \frac{3}{32\pi} \frac{hc^3}{k} \frac{\lambda_{21}^2}{c^2} A_{21} N(HI)$$

$$\boxed{\int T_A - T_A(0)\, dv = 54.89\ K \cdot km/s\ \frac{N(HI)}{10^{20}\ cm^{-2}}}$$

$\Rightarrow$ Integrated line intensity tells us $N(HI)$ without need to know T_S.

Total Mass: $M_{HI} = 4\pi D_L^2 \cdot N(HI)\, m_H$

$$= 4\pi D_L^2 \cdot m_H \cdot \frac{16\pi}{3} \frac{1}{A_{21} h\nu_{21}} \int I_\nu - I_\nu(0)\, d\nu$$

$F_{obs} = 4\pi$ steradians $\times \int I - I_\nu(0)\, d\nu$

see also 8.26–8.28

$$M_{HI} = \frac{16\pi m_H}{3 A_{21} h\nu_{21}} D_L^2 F_{obs}$$

$$= 4.945 \times 10^7 M_\odot \left(\frac{D_L}{Mpc}\right)^2 \left(\frac{F_{obs}}{Jy\ MHz}\right)$$

$d\nu_{obs} = \frac{\nu_{21}}{1+z}\frac{dv}{c}$

$$= 2.343 \times 10^5 M_\odot (1+z)(D_L/Mpc)^2 \left(\frac{\int F_\nu dv}{Jy\ km/s}\right)$$