UNIVERSITY OF CALIFORNIA, SANTA BARBARA
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Physics 233 **Exercise 4 (Due Wed. Feb. 19)** Winter 2014

Ionization Structure of a Hydrogen Nebula

Beginning with the ionization equilibrium equation, written in the on-the-spot approximation, show that the degree of ionization, $\chi \equiv n(H^+)/n(H)$, of a pure hydrogen gas cloud surrounding a star which radiates like a black body can be written as

$$\frac{\chi^2}{1-\chi} = \frac{2\pi}{c^2}\frac{R_*^2}{r^2}\frac{1}{n(H)\alpha_B(T_e)}\int_{\nu_0}^{\infty}\nu^2(e^{h\nu/kT_*}-1)^{-1}e^{-\tau_\nu}\sigma(\nu)d\nu$$

where

$$\tau_\nu = n(H)\sigma(\nu)\int_{R_*}^{r}(1-\chi)dr.$$

This equation would be a simple quadratic to be solved for χ in terms of distance from the star, r, were it not for the dependence of optical depth on χ. In order to solve this ionization equation for a nebula, one must solve for χ in terms of r in discrete steps as follows.

Divide the H II region into about 100 concentric shells of width Δr. Assume that the ionization is constant within each of the shells, but that it changes from shell to shell. This assumption is not too bad, since χ changes slowly everywhere except near the transition region. Denote the inner radius of the i-th shell as r_i and the ionization of that shell as χ_i. Begin the calculation at the surface of the star, $r_1 = R_*$. In this shell, $\tau_\nu = 0$, and the integral can be evaluated easily by Gauss-Laguerre quadrature (for example). The ionization is then obtained by solving the quadratic equation for χ_1.

In order to calculate χ_2, you need to know $\tau_\nu(r_2)$; but this is

$$\tau_\nu(r_2) = n(H)\sigma(\nu)(1-\chi_1)\Delta r\ .$$

Use this in the integral, and evaluate it at r_2 by, for example, numerical quadrature. Then solve the quadratic equation again, for χ_2. Repeat this procedure until $\chi \to 0$, i.e., solve for χ_i at each r_i by setting

$$\tau_\nu(r_i) = \Sigma_{j=1}^{i-1} n(H)\sigma(\nu)(1-\chi_j)\Delta r\ .$$

Write a program on the computer of your choice to determine $\chi(r)$ under the following conditions. Let $n(H) = 1\,\mathrm{cm}^{-3}$, and

$$\alpha_B = \Sigma_{n=2}^{\infty} < \sigma_n^{rec} v > = 2.6\times 10^{-13}\mathrm{cm}^3\mathrm{sec}^{-1}$$

as is true for H at $T_e = 10{,}000$ K. Take the absorption cross-section to be

$$\sigma(\nu) = 6.3\times 10^{-18}\Big(\frac{\nu_0}{\nu}\Big)^3\mathrm{cm}^2\ .$$

Evaluate the integral. Change variables so that the limits of integration go from $0 \to \infty$,

$$\int_0^\infty f(x)e^{-x}dx = \Sigma_i\, a_i f(x_i).$$

(You can then, for example, use Gaussian quadrature with an adequate number of points (say 8 or so); see for example Abromowitz and Stegun, p. 916, 923.)

Guess the step size to use in the problem, by calculating the expected size of the H II region using the analytical expression for the Stromgren radius derived in class:

$$R_S = 78 \text{ pc } n(H)^{-2/3} \left(\frac{R_*}{R_\odot}\right)^{2/3} \frac{F_1^{1/3}(x_1)}{x_1},$$

where $x_1 \equiv \frac{h\nu_0}{kT_*}$. Take a step size of about $1\% R_S$. Assume the star has $T_* = 40,000$ K, and $R_* = 10\, R_\odot$. Print out (and plot) the following quantities at each r_i : $(1 - \chi_i)$, χ_i, and $\tau_{\nu_0}(r_i)$, the optical depth at the Lyman limit.

SUPPLEMENTARY INFORMATION

The Integral $F_1(x_1)$

$T_*(^\circ K)$	x_1	$F_1(x_1)$
10,000	15.79	3.91×10^{-5}
15,000	10.53	3.58×10^{-3}
20,000	7.90	2.98×10^{-2}
25,000	6.32	9.85×10^{-2}
30,000	5.26	0.209
35,000	4.51	0.346
40,000	3.95	0.495
45,000	3.51	0.645
50,000	3.16	0.790
60,000	2.63	1.05
70,000	2.26	1.26
80,000	1.97	1.43
90,000	1.75	1.57
100,000	1.58	1.69